5. Oscillation

5.1.

Theory

Harmonic oscillation is basically dealt with in various fields – not only in classical mechanics but also in quantum mechanics, acoustics, thermodynamics, electromagnetic and so on. Mechanical oscillation represents the repetition of motion and electric oscillation represents the change of voltage and electric current. Harmonic oscillation is important because in case of small oscillations in the system equilibrium, the solution[1] can be sought perfectly and analyzed, and although the potential is not consistent generally, it can be dealt with as harmonic oscillation near the minimum within the potential well. Classically, it cannot pass through the wall of the well, so the oscillation occurs within the potential, and you can check this out by simulations or experiments.

In a physical situation, one-dimensional potential is the base for explaining harmonic oscillations. Let's understand the one-dimensional potential of harmonic oscillation theoretically by representing it with the system of mass and spring, and examine it with simulations and actual experiments.

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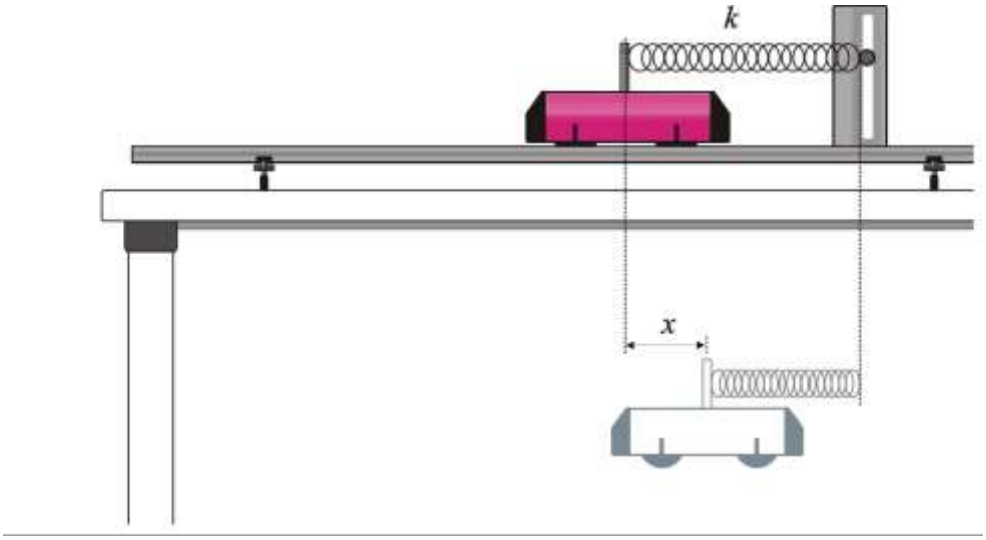
[1] In classical mechanics, when Hamiltonian is equal to the total energy of a system, general solutions of location or velocity can be attained by Hamilton's equation of motion according to the flow of time.



Picture 5.1.1 Harmonic oscillation of the system of mass-spring[2]: Oscillation experiment composed of a cart and a spring

5.1.1. Free Oscillation and Damped Oscillation

$$V(x,t)=\frac{1}{2}k\dot{x}^2$$

When the mass m moves within the potential well $V(x,t)$ and the resistance of the system is small, Lagrangian can be shown like below.

$$L(x_i,\dot{x}_i,t)=T-V=\frac{1}{2}m\dot{x}^2-V(x,t) \qquad (5.1.1)$$

And the general form of Lagrange equation of motion represented by generalized coordinate and velocity is as follows.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}=0$$

[2] Various experiment designs are possible according to how to compose the mass and the spring, and the equation of motion can be set up complying with each composition.

If you substitute formula (5.1.1) and solve this, the result will be like below.

$$\frac{\partial L}{\partial x} = \frac{\partial\,(T-V)}{\partial\,x} = \frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) = -kx$$

$$\frac{\partial\,L}{\partial\,\dot{x}} = \frac{\partial\,(T-V)}{\partial\,\dot{x}} = \frac{\partial}{\partial\,\dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = m\dot{x}$$

Therefore, you can get the equation of motion below.

$$\frac{d}{dt}\frac{\partial\,L}{\partial\,\dot{x}} - \frac{\partial\,L}{\partial\,x} = \frac{d}{dt}(m\dot{x}) + kx = 0$$

$$\therefore\ m\ddot{x} + kx = 0 \qquad (5.1.2)$$

Formula (5.1.2) shows Newton's equation of motion is applicable. This explains that potential energy is not the function of time but the most general equation of motion of one-dimensional motion[3].

Formula (5.1.2) can be rewritten when $w_0^2 = \frac{k}{m}$.

$$\ddot{x} + \omega_0^2 x = 0 \qquad (5.1.3)$$

In Hamilton's equation, the momentum is like below.

$$p = \frac{\partial\,L}{\partial\,\dot{x}} = m\dot{x}$$

So, the energy is as follows.

$$H = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}\left(\frac{p}{m}\right)^2 + \frac{1}{2}kx^2$$

.

[3] A form of motion whose degree of freedom is 1

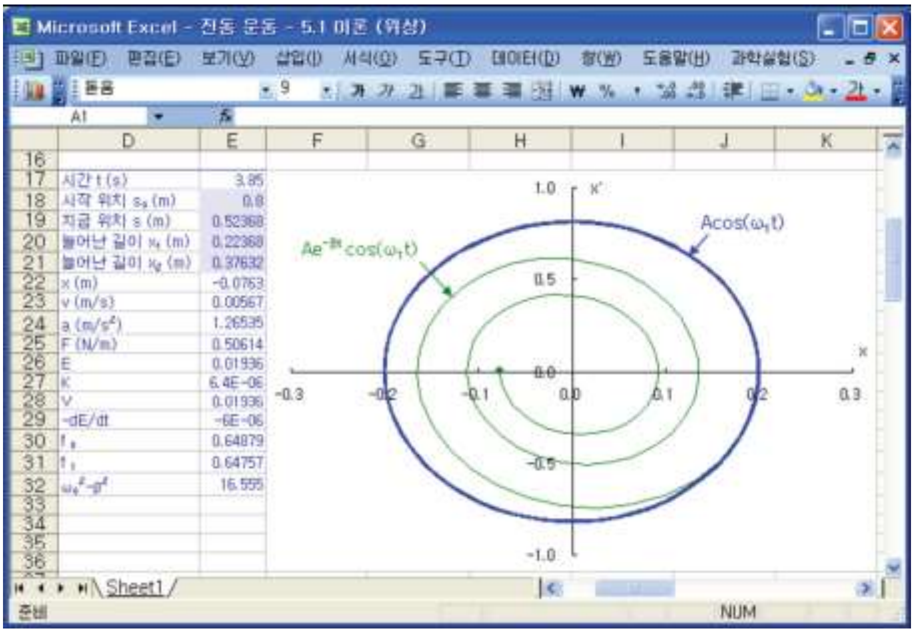
Change Hamiltonian H to E, divide both sides with E and rewrite the formula in the form of the equation of ellipse. Then it will be changed into the formula about generalized coordinate x and the phase-space of momentum P below.

$$\frac{x^2}{2E/k}+\frac{\dot{x}^2}{2E/m}=\frac{x^2}{2E/k}+\frac{p^2}{2mE}=1 \qquad (5.1.4)$$

In formula (5.1.4), the amplitudes of axes will be $\sqrt{2E/k}$ (in case of x) and $\sqrt{emE}$ (in case of p). If you multiply the radiuses of the ellipse's long axis and short axis, the result is like below.

$$a_n b_n = \sqrt{\frac{2E}{m}} \cdot \sqrt{\frac{2E}{k}} = 2E\left(\frac{1}{\sqrt{mk}}\right)$$

This represents that the energy level is widened in a regular interval when the area of the ellipse is quantized classically. More specified information will be stated in Quantum Mechanics. If the phase-space of x and $\dot{x}$ is expressed by the simulation in Excel, the result will be same as picture 5.1.2.



Picture 5.1.2 Phase-space of x and $\dot{x}$ expressed by the simulation in Excel: the motion of $A\cos\omega_1 t$ is when the attenuation constant $\beta = 0$.

Until now, the equations of motion have been solved on the assumption that the experiment is done in vacuum where the resistance of medium does not exist. In fact, in case of motion, the medium resists to disturb the motion of an object. The motion of an object gets damped and after a while it stops. When there is no outer force and the frictional force of a system composed of mass and a spring is $-b\dot{x}$, the equation of motion is as follows.

$$m\ddot{x}+b\dot{x}+kx=0 \qquad (5.1.5)$$

Like this, 선형 동차 방정식 becomes secondary differential equation that has a constant as a coefficient. And the general solution of it should be calculated by exponential function. If we rewrite the formula (5.1.5) when $\beta=\frac{b}{2m}, \omega_0=\sqrt{\frac{k}{m}}$, the result is as follows.

$$\ddot{x}+2\beta\dot{x}+\omega_0^2 x=0 \qquad (5.1.6)$$

There is a solution in the form of $x=Ae^{\pm\gamma t}$. Substitute $x=Ae^{\pm\gamma t}$ to formula (5.1.5), and the result will be like below.

$$\gamma^2 Ae^{-\gamma t}+2\beta(-\gamma Ae^{-\gamma t})+\omega_0^2(Ae^{-\gamma t})=0$$

Two solutions of γ will be

$$\gamma=\frac{-2\beta\pm\sqrt{(-2\beta)^2-4\omega_0^2}}{2}=-\beta\pm\sqrt{\beta^2-\omega_0^2}$$

And the general solution of formula (5.1.6) is shown below.

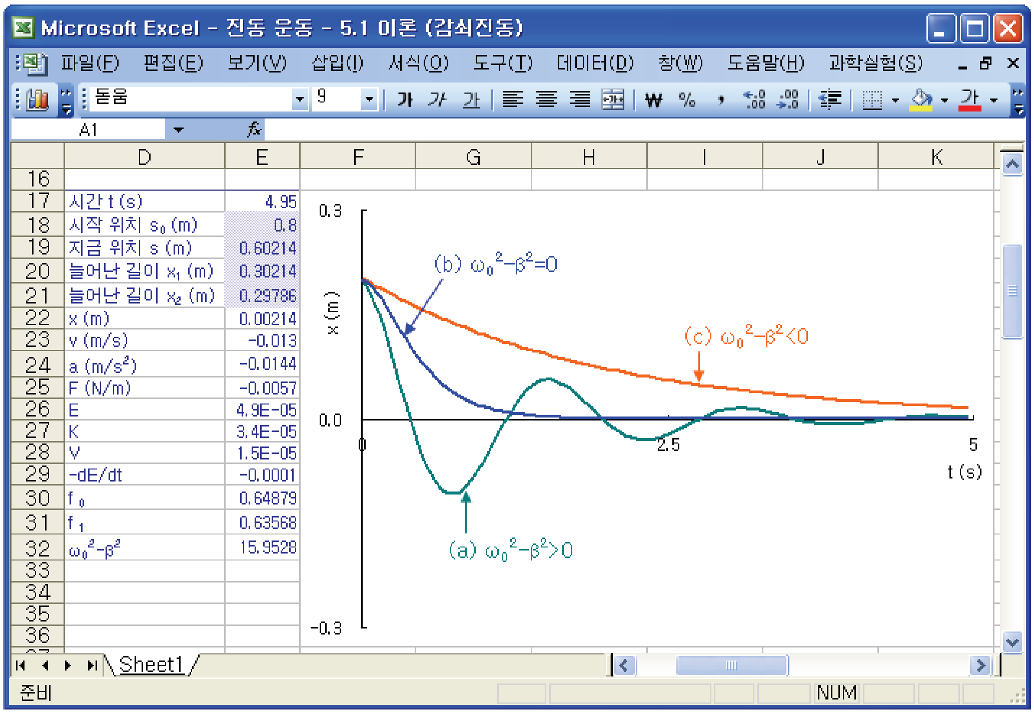
$$x=e^{-\beta t}\left[c_1 e^{\sqrt{\beta^2-\omega_0^2}\,t}+c_2 e^{t\sqrt{\beta^2-\omega_0^2}\,t}\right] \qquad (5.1.7)$$

In formula (5.1.7) three general cases below can be considered.

(a) $\omega_0^2 - \beta^2 > 0$: Minute damped oscillation

(b) $\omega_0^2 - \beta^2 = 0$: Critical damped oscillation

(c) $\omega_0^2 - \beta^2 < 0$: Excessive damped oscillation



Picture 5.1.3 Damped motion expressed by the simulation in Excel: In the simulation, if the mass is 0.4 kg and modulus of elasticity is 6.647, the attenuation constant of the critical damped motion b=3.26116.

(a) In case of $w_0 = \beta$, when $\omega_1^2 = \omega_0^2 - \beta^2$, formula (5.1.7) can be changed.

$$x = e^{-\beta t}\left[c_1 e^{i\omega_1 t} + c_2 e^{-i\omega_1 t}\right]$$

And this formula can be completed as follows.

$$x = Ae^{-\beta t}\cos(\omega_1 t + \theta) \qquad (5.1.8)$$

In case of motion according to formula (5.1.8), it takes the form of a subtle underdamped harmonic motion as time goes by. In this case, this envelope motion is the amplitude of this equation of motion which is represented as a cos function.

$$x_A = Ae^{-\beta t} \qquad (5.1.9)$$

In case of motion (a), it can be checked out by simulations and experiments in Excel.

In case of (b) $w_0 = \beta$, $b = 2\sqrt{mk}$ and a critically damped motion occurs. The general solution of it is as follows.

$$x = (A_1 + A_2 t)e^{-\beta t} \qquad (5.1.10)$$

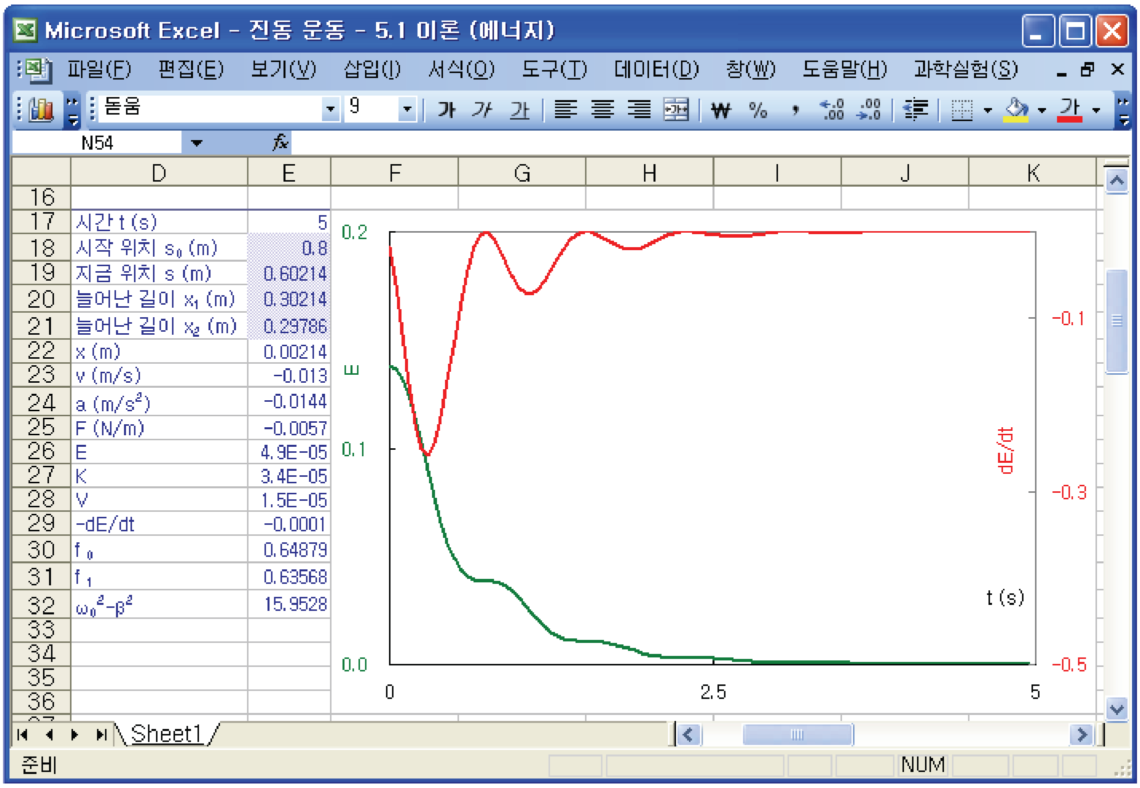
In case of (c) $\omega_0 < 0$, the frictional resistance is so big that an overdamped motion occurs. If $\omega^2 = \beta^2 - \omega_0^2$, the general solution is like below. In this case, ω is not the angular velocity representing a real periodic motion but the constant about exponential damping.

$$x = e^{-\beta t}\left[A_1 e^{\omega t} + A_2 e^{-\omega t}\right] \qquad (5.1.11)$$

If the location and velocity is calculated as the general solutions of the equation of motion, you can understand the mechanical energy which is the sum of a system's energy and the ratio of energy decrease per hour(dE/dt). dE/dt is as follows.

$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial x}{\partial t}\frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2\right) + \frac{\partial x}{\partial t}\frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) \\ &= (-b\dot{x})\dot{x} \\ &= -b\dot{x}^2\end{aligned}$$

Picture 5.1.4 is the graph representing a system's mechanical energy and the ratio of energy decrease per hour.

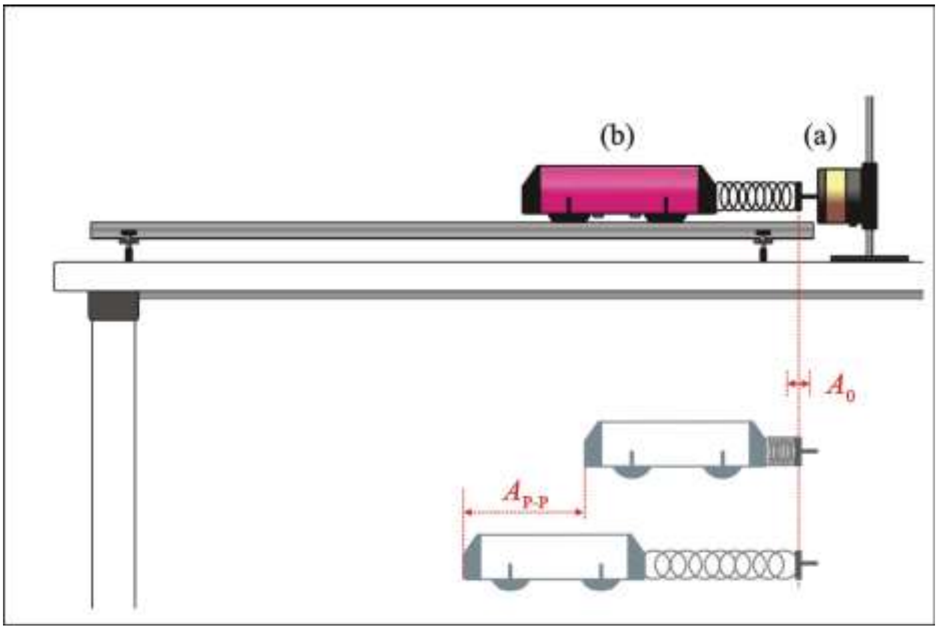


Picture 5.1.4 Mechanical energy and the ratio of energy decrease per hour

With reference to the explanation about the damped motion, try to solve formula (5.1.7) in case of (b) and (c). In 5.1.2, forced vibration is explained. 5.2 includes the process of calculating the physical value about motions of systems (amplitude, oscillation frequency, velocity, energy and so on) in a damped oscillation. It is done by simulations using Excel. The simulation is a process in which the physical concepts and knowledge are solved by computer simulation[4]. Through the process of simulations, the concepts which were difficult to understand can be understood well. This simulation processes can be conducted optionally before the experiment.

[4] Simulation is a mathematical solving process of physical theories and there is professional software which can conduct simulations. However, this book indicates that conducting simulations is easily possible in Excel.

5.1.2. Forced Vibration



Picture5.1.5 Forced vibration experiment of a cart in a damped motion: (a) mechanical waver driver[5] (b) cart (Forced harmonic oscillator: FHO)

Let's find out about forced harmonic motion, in which the FHO oscillates by getting a periodical force such as a sinusoidal force from outside. The equation of motion of it is as follows.

$$m\ddot{x}+kx+b\dot{x}=F_0\cos\omega t \qquad (5.1.12)$$

Just like the right term of formula (5.1.12), the forced vibration frequency is prominent when it reaches near the natural oscillation frequency of the harmonic oscillator. In contrast to the damped motion, the amplitude of the oscillator increases enormously in this oscillation, and this is called resonance.

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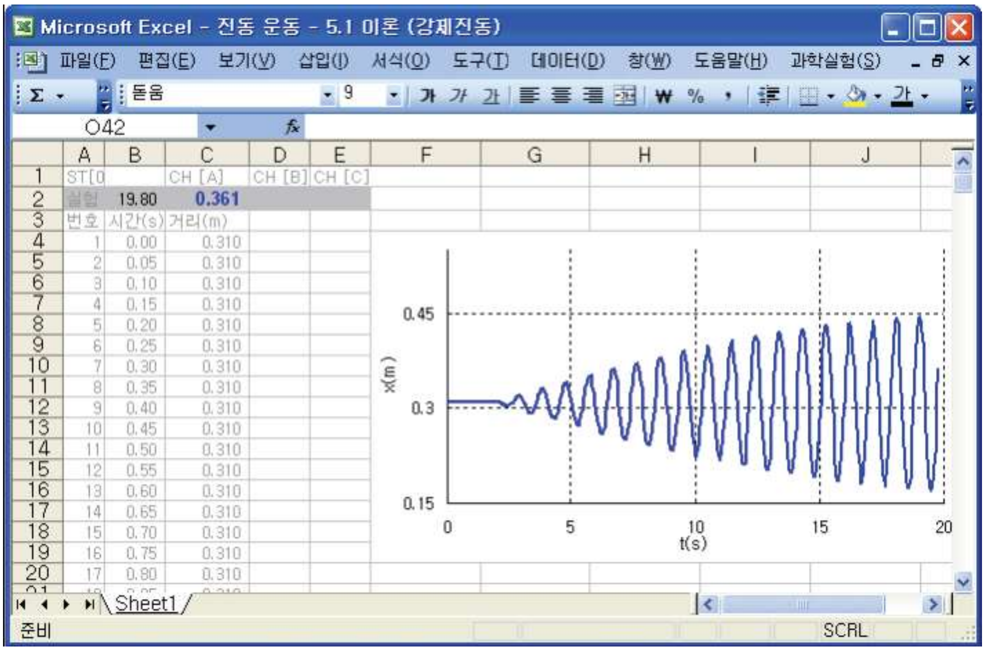
[5] Mechanical waver driver oscillates by receiving signals of $\sin$ function from the function generator.



Picture 5.1.6 Forced vibration of a system which oscillates near resonance: The amplitude reaches at the maximum.[6]

When FHO gets a periodical force which is dependent on time, it passes the transient state and oscillates in the steady-state. For a short time, this transient oscillation and forced vibration are superposed in a linear form. The state of forced vibration has two states like this, and the steady-state is dependent on time, so solve it by calculating the solution of the in homogeneous equation. Formula (5.1.12) is the real part of forced vibration motion. The general solution of the equation of motion about FHO which includes the two states is like below.

$$x = A_D e^{-\beta t}\cos(\omega t + \phi_D) + A_\omega \cos(\omega t + \phi) \qquad (5.1.13)$$

The first term of formula (5.1.13) disappears as time goes by and the second term which is the real part of $e^{i\omega t}$ remains only, which is the solution of the steady-state. Calculate the amplitude $A(\omega)$ using this solution.

[6] If the amplitude increases immensely, the system can be broken and in the mechanic system, this situation is needed or not according to the circumstances.

Express x as the complex exponential function.

$$x = Ae^{(\omega t \pm \phi)} \qquad (5.1.14)$$

Substitute this to formula (5.1.13) and divide both sides with $e^{i\omega t}$.

$$\begin{aligned} -m\omega^2 A + ib\omega A + kA &= F_0 e^{-i\phi} \\ &= F_0(\cos\phi - i\sin\phi) \end{aligned} \qquad (5.1.15)$$

Divide the real part and the imaginary part in formula (5.1.15).

$$\begin{aligned} A(k - m\omega^2) &= F_0\cos\phi \\ -b\omega A &= F_0\sin\phi \end{aligned} \qquad (5.1.16)$$

Square both sides of formula (5.1.16) and add.

$$A^2(k - m\omega^2)^2 + b^2\omega^2 A^2 = (F_0\cos\phi)^2 + (-F_0\sin\phi)^2$$

Substitute $\omega_0^2 = k/m$, $\beta = b/2m$ and calculate the amplitude $A(\omega)$ about the oscillation frequency (ω) of forced vibration.

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \qquad (5.1.17)$$

Take $\tan\phi$ from formula (5.1.16) and calculate phase ϕ.

$$\phi = -\tan^{-1}\left[\frac{\beta\omega/m}{\omega_0^2 - \omega^2}\right] \qquad (5.1.18)$$

Let’s consider formula (5.1.17) and (5.1.18) in following ways.

(a) If $\omega \langle\langle \omega_0$ and ω is so small ($\omega \approx 0$) that the phase ϕ reaches near 0, then the amplitude $A = A_0(\omega = 0)$ is as follows.

$$A_0 = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k} \qquad (5.1.19)$$

$$\therefore F_0 = kA_0$$

This shows that if the oscillation frequency is small, just like in the free oscillation, it becomes the amplitude A_0 when F_0 force is operated,

(b) If $\omega \rangle\rangle \omega_0$ and ω is big, the denominator of formula (5.1.17) becomes like this: $\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \approx \omega^2$. Therefore the amplitude A is like below.

$$A(\omega \gg \omega_0) \simeq \frac{F_0}{m\omega^2} = (A_0\omega_0) \cdot (1/\omega^2) = \frac{\omega_0^2}{\omega^2} A_0 \qquad (5.1.20)$$

In this formula, the phase ϕ reaches near π the amplitude decreases as much as the oscillation frequency increases. Eventually it becomes $1/\omega^2$.

(c) This is about $\omega \approx \omega_0$. If the damping becomes 0 so it becomes like $\omega \cong \omega_0$, then $\omega_R \cong \omega_D \cong \omega_0$. The phase passes $1/2\pi$ at ω_0 . However, experimentally[7], the damping cannot be 0 so the resonance frequency is definitely not equal to ω_0. In formula (5.1.17), when the amplitude A becomes the maximum at a certain moment, the resonance frequency ω_R is as follows.

$$\omega_R^2 = \omega_0^2 - 2\beta^2 = \omega_D^2 - \beta^2 \qquad (5.1.21)$$

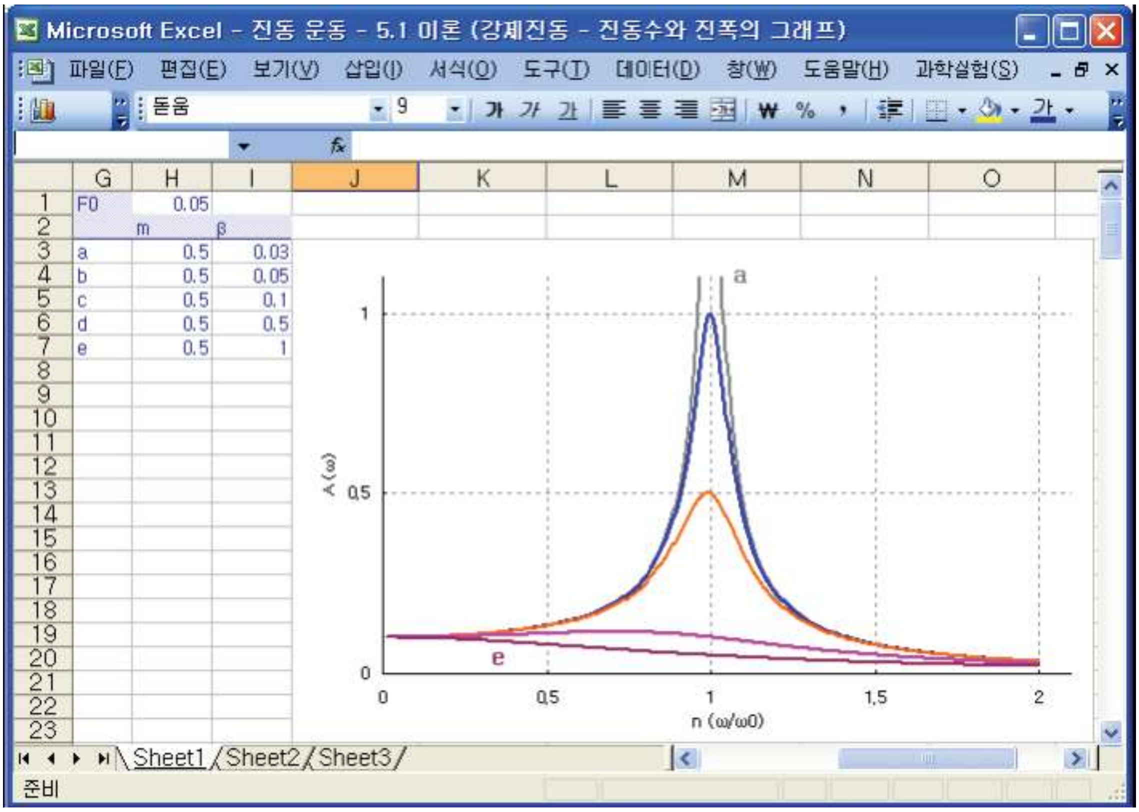
If the damping is really weak, in formula (5.1.17) when the damping constant is very small, the maximum amplitude near resonance becomes as follows.

[7] When doing the forced vibration experiment, analyze ω_0 , ω_D in case of the damped motion and examine the amplitude in the circumstance of forced vibration $n = \omega/\omega_0$.

$$A_R \simeq \frac{F_0}{2\beta m\omega_0} \qquad (5.1.22)$$

In formula (5.1.22), the resonance becomes the maximum as the damping gets smaller and reaches 0. If you divide formula (5.1.19) with formula (5.1.22), you can calculate the Q constant.

$$Q = \frac{A_R}{A_0} = \frac{F_0/2\beta m\omega_0}{F_0/m\omega^2} = \frac{\omega_0}{2\beta} = \omega_0\tau \qquad (5.1.23)$$



Picture 5.1.7 The graph of the amplitude $A(\omega)$ and the oscillation frequency $n = \omega/\omega_0$ [8]

[8] This graph is drawn in Excel using formula (5.1.17). You can find out the factors affecting the forced vibration by changing the force F_0, mass m, the attenuation constant β and making various graphs.

Picture 5.1.7 is a graph drawn from using formula (5.1.17) and changing the attenuation constant β. When the damping is small and the resonance occurs in the steady-state, the displacement, velocity and force can be calculated like below by solving the real part of $x = Ae^{i(\omega t-\phi)}$.

$$x_R = A_0 \sin\omega_0 t$$
$$\dot{x}_R = \omega_0 A_0 \cos\omega_0 t \qquad (5.1.24)$$
$$F = F_0 \cos\omega_0 t$$

In formula (5.1.24), F_0 can be calculated as $F_0 = k[A_{0(p-p}/2]$ by estimating the amplitude $A_{0(p-p)}$ from the mechanical wave driver. At this time, the natural angular frequency ω_0 is calculated as $\sqrt{k/m}$ (m is the mass of a cart and k is the modulus of elasticity). If the damping resistance is small, it is almost the same as the value of damped oscillation[9], so you can get the resonant oscillation experimentally near this value. Also, the result can be set up as the condition and the target value for finding solutions in physical modeling, so it can create the mathematical prediction model for oscillations. Set up the physical model of the oscillation motion according to the data analysis based upon the physical modeling, and predict the experimental circumstances. Also, research the character of oscillation motion by solution finding and curve fitting.

[9] This is the free running state which only has the damped oscillation.

5.2.

Simulation

Oscillation motion modeling consisted of mass-spring can be realized as simulations by using the general solution of linear homogeneous equation of motion or by solving it with linear secondary ODE(ordinary differential equation). If the results of mathematical solution represent that improper model has been chosen, then there will be difficulties of representing physical states which are different with the real experiment results. When learning physics, this situation can be challenges for the students. Therefore, the experiments should be designed delicately and the results should be compared carefully[10].

The simple and general way to realize the one dimensional harmonic oscillation by simulations in Excel is to get the mathematical formula concerning basic physical quantities such as location, velocity and so on. This can be done by using the general solution of the equation of motion. On the other hand, the location and velocity of the mass can be calculated by ODE. If the location and velocity is calculated, other physical quantities concerning the motion of a system can be calculated, too.

[10] Simulation is an effective tool for understanding the concepts of physics, but in this book, it can be omitted according to the circumstances.

5.2.1. General Solution Solving Process

The way of using the general solution is to calculate the location and velocity with the function of time. In case of formula (5.1.7), which is the equation of motion concerning a system with mass-spring, let's calculate the displacement x and velocity $\dot{x}$ when (a) $\omega_0^2 - \beta^2 > 0$, (b) $\omega_0^2 - \beta^2 = 0$, (c) $\omega_0^2 - \beta^2 < 0$.

(a) When $\omega_0^2 - \beta^2 > 0$, if the velocity $\dot{x}$ is calculated by setting the initial condition $\theta = 0$ in formula (5.1.8), the result is as follows.

$$\dot{x} = -A\beta e^{-\beta t}\cos\omega_1 t - A\omega_1 e^{-\beta t}\sin\omega_1 t$$

Substitute x of formula (5.1.8).

$$\dot{x} = -\beta x - A\omega_1 e^{-\beta t}\sin\omega_1 t \qquad (5.2.1)$$

(b) When $\omega_0^2 - \beta^2 = 0$, if the velocity $\dot{x}$ is calculated by setting the initial condition $t_0 = 0, x(t_0) = A, \dot{x}(t_0) = 0$ in formula (5.1.10), the result is like below.

$$x(t_0) = (A_1 + A_2 t_0)e^{-\beta t_0}$$

$$A = (A_1 + 0)e^{-0}$$

$$\therefore A_1 = A$$

Substitute $A_0 = A$ for formula (5.1.10) and differentiate it.

$$\dot{x} = A_2 e^{-\beta t} - \beta e^{-\beta t}(A + A_2) = A_2 e^{-\beta t} - \beta x$$

$$0 = A_2 e^{-\beta \cdot 0} - \beta \cdot A$$

$$\therefore A_2 = \beta A$$

If $A_1 = A, A_2 = \beta A$ is substituted for formula (5.1.10), the result is as follows.

$$x = (A_1 + A_2 t)e^{-\beta t}$$
$$= (A + \beta A t)e - \beta t$$
$$\therefore x = (1 + \beta t)Ae^{-\beta t} \qquad (5.2.2)$$

The result of differentiating formula (5.2.2) is like below.

$$\dot{x} = \beta A e^{-\beta t} - \beta A(1 + \beta t)e^{-\beta t}$$

So the velocity $\dot{x}$ is as follows.

$$\dot{x} = \beta(Ae^{-\beta t} - x) \qquad (5.2.3)$$

(c) When $\omega_0^2 - \beta^2 < 0$, if the velocity $\dot{x}$ is calculated by setting the initial condition $t_0 = 0, x(t_0) = A, \dot{x}(t_0) = 0$ in formula (5.1.11), the result is like below.

$$x(t_0) = e^{-\beta t_0}\left[A_1 e^{\omega t_0} + A_2 e^{-\omega t_0}\right]$$
$$A = e^{-\beta \cdot 0}\left[A_1 e^{\omega \cdot 0} + A_2 e^{-\omega \cdot 0}\right]$$
$$\therefore A_2 = A - A_1$$

The result of differentiating formula (5.1.11) and substituting the initial condition is as follows.

$$\dot{x} = \omega_2\left[A_1 e^{\omega t} - (A - A_1)e^{-\omega t}\right]e^{-\beta t} - \beta x$$
$$0 = \omega\left[A_1 e^0 - (A - A_1)e^{-0}\right]e^{-0} - \beta A$$
$$= \omega\left[A_1 - A + A_1\right] - \beta A$$
$$= \omega\left[2A_1 - A\right] - BA$$

Solve the formula above by A_1.

$$\beta A = \omega[2A_1 - A] = 2\omega A_1 - \omega A$$

$$2\omega A_1 = \beta A + \omega A = (\beta + \omega)A$$

$$\therefore A_1 = \left(\frac{\beta + \omega}{2\omega}\right)A$$

So, if $A_2 = A - A_1, A_1 = (\frac{\beta + \omega}{2\omega})A$ is substituted for formula (5.1.11), the result is as follows.

$$x = e^{-\beta t}\left[\left(\frac{\beta + \omega}{2\omega}A\right)e^{\omega t} + \left(A - \frac{\beta + \omega}{2\omega}A\right)e^{-\beta t}\right]$$

$$\therefore x = Ae^{-\beta t}\left[\left(\frac{\beta + \omega}{2\omega}\right)e^{\omega t} + \left(1 - \frac{\beta + \omega}{2\omega}\right)e^{-\beta t}\right] \qquad (5.2.4)$$

Calculate the velocity $\dot{x}$ by differentiating formula (5.2.2).

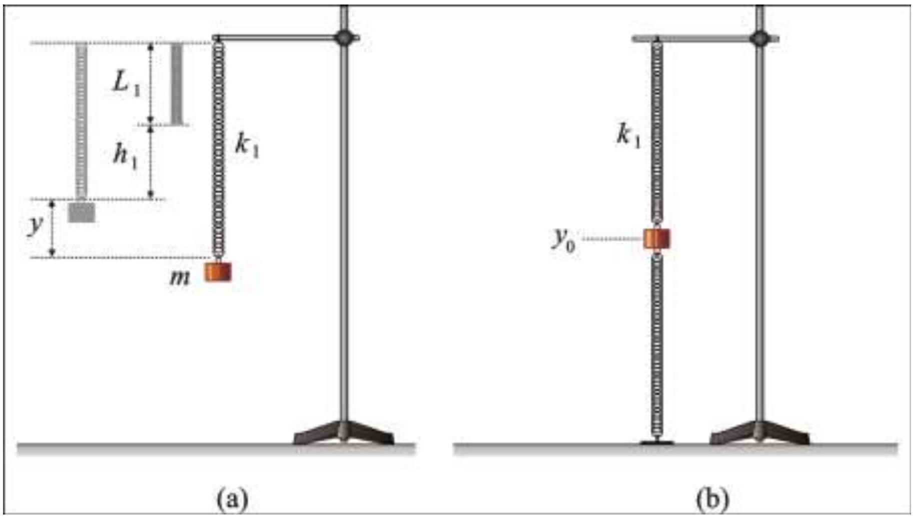
$$\dot{x} = \omega\left[\left(\frac{\beta + \omega}{2\omega}\right)Ae^{\omega t} - \left(A - \frac{\beta + \omega}{2\omega}A\right)e^{-\omega t}\right] \cdot$$
$$e^{-\beta t} - \beta x$$

$$\therefore \dot{x} = A\omega e^{-\beta t}\left[\left(\frac{\beta + \omega}{2\omega}\right)e^{\omega t} - \left(1 - \frac{\beta + \omega}{2\omega}\right)e^{-\omega t}\right] \qquad (5.2.5)$$
$$- \beta x$$

Formula 5.2.1, 5.2.2, 5.2.3, 5.2.4, 5.2.5, which is about the displacement x and velocity $\dot{x}$ can be used during the simulations. The results calculated by these formulae can be compared within the error range caused by the uncertainty of the estimation and can be used to analyze and predict the results of experiments done by the physical models.

Exercise 5.2.1: The Location and Velocity of Spring Pendulum

Calculate the general solution of spring pendulums like picture 5.2.1 and express in formulae in case of $\omega_0^2 - \beta^2 > 0$, $\omega_0^2 - \beta^2 = 0$, $\omega_0^2 - \beta^2 < 0$.



Picture 5.2.1 the oscillation of spring pendulums

Explanation:

In case of (a), the equation of motion is like below.

$$m\ddot{y} + \beta\dot{y} + ky - mg = 0$$

And the general solution is as follows.

$$y = h_1 + Ae^{-\beta t}\cos\omega t$$

Therefore, in case of $\omega_0^2 - \beta^2 > 0$, $\omega_0^2 - \beta^2 = 0$, $\omega_0^2 - \beta^2 < 0$,

(ㄱ) When $\omega_0^2 - \beta^2 > 0$, y and $\dot{y}$ are as follows.

$$y = h_1 + Ae^{-\beta t}\cos\omega t$$
$$\dot{y} = -\beta(y - h_1) - A\omega e^{-\beta t}\sin\omega t$$

(ㄴ) When $\omega_0^2 - \beta^2 = 0$, y and $\dot{y}$ are as follows.

$$y = h_1 + A(1 + \beta t)e^{-rt}$$
$$\dot{y} = \beta\left[Ae^{-\beta t} - (y - h_1)\right]$$

(ㄷ) When $\omega_0^2 - \beta^2 < 0$, y and $\dot{y}$ are as follows.

$$y = h_1 + Ae^{-\beta t}\left[\left(\frac{\beta + \omega}{2\omega}\right)e^{\omega t} + \left(1 - \frac{\beta + \omega}{2\omega}\right)e^{-\beta t}\right]$$
$$\dot{y} = A\omega e^{-\beta t}\left[\left(\frac{\beta + \omega}{2\omega}\right)e^{\omega t} - \left(1 - \frac{\beta + \omega}{2\omega}\right)e^{-\omega t}\right] - \beta x$$

In case of (b), the elastic force of the whole mass-spring system becomes $-(k_1 + k_2)x$, so $k = k_1 + k_2$ and the equation of motion is like below.

$$m\ddot{y} + \beta\dot{y} + ky - mg = 0$$

Therefore, you can solve the cases of $\omega_0^2 - \beta^2 > 0$, $\omega_0^2 - \beta^2 = 0$, $\omega_0^2 - \beta^2 < 0$ in the same way by changing the modulus of elasticity from (a).

5.2.2. ODE Solving Process

The way to solve linear secondary ODE is to calculate not the general solution but the location and velocity per hour directly. The result of this solution can be compared to the result of simulation during the general solution solving process. The equation of motion about a mass-spring system is as follows.

$$m\ddot{x} + kx + b\dot{x} = 0$$

Solve this formula by $\ddot{x}$.

$$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x} = -2\beta\dot{x} - \omega_0^2 x$$

This equation includes the secondary differential term of x, so solve it by dividing it into two linear ODE.

ODE uses the 4th RK (Runge-Kutta) way, which has high accuracy and can calculate the location and velocity with certain interval. If the equation of motion is divided into two formulae, the result is like below.

$$\begin{aligned} f(t,\ x) &= \dot{x} = v \\ f(t,\ x,\ v) &= \ddot{x} = -2\beta v - \omega_0^2 x \end{aligned} \qquad (5.2.6)$$

Apply RK here. RK is a solution of the secondary ODE, which has next 4 steps.

$$\begin{aligned} x_a &= hv \\ v_a &= hf(t,\ x, v) \end{aligned}$$

When there is a small increase of change(h) which has constant interval, consider the location x_a and v_a as the first formula above.

$$x_b = h(v + v_a \cdot \frac{1}{2})$$
$$v_b = hf(t + \frac{h}{2,} x + \frac{x_a}{2},\ v + \frac{v_a}{2})$$

When the change increases and reaches the middle $\frac{h}{2}$, the gradient of x, that is, v can be calculated as below.

$$x_c = h(v + v_b \cdot \frac{1}{2})$$
$$v_c = hf(t + \frac{h}{2}, x + \frac{x_b}{2}, v + \frac{v_b}{2})$$

$$x_c = h(v + v_b \cdot \frac{1}{2})$$
$$v_c = hf(t + \frac{h}{2}, x + \frac{x_b}{2}, v + \frac{v_b}{2})$$

Lastly, the 4th RK formula about this equation of motion is as follows.

$$x_d = h(v + v_c)$$
$$v_d = hf(t + h, x + x_c, v + v_c)$$

Here, substitute the time interval dt for the small change h. Consequently, the optimum solution[11] for x and v is as follows.

$$x += (x_a + 2x_b + 2x_c + x_d)/6$$
$$v += (v_a + 2v_b + 2v_c + v_d)/6 \qquad (5.2.7)$$

[11] When doing the simulation with formula 5.2.7, the values of a, b, c, d processes were used all. The formula of RK, which is like formula 5.2.7, was used to calculate the impulse in Chapter 4 Collisions.

Exercise 5.2.2 ODE of a Spring Pendulum

Solve ODE about the motion of a spring pendulum in picture 5.2.1 which has periodical damping.

<u>Explanation:</u>

The formula of RK used to solve ODE generally has the form[12] as below.

$$f(h+1)=f(h)+(K_1+2K_2+2K_3+K4)h/6 \qquad (5.2.8)$$

In oscillations, the solution of the velocity for the equation of motion was calculated by formula (5.2.1), so formula (5.2.6) can be rewritten like below.

$$f(t,\ x)=\dot{x}=v$$
$$f(t,\ x,\ \dot{x})=\ddot{x}=-2\beta\dot{x}-\omega_0^2(x-h_1)$$

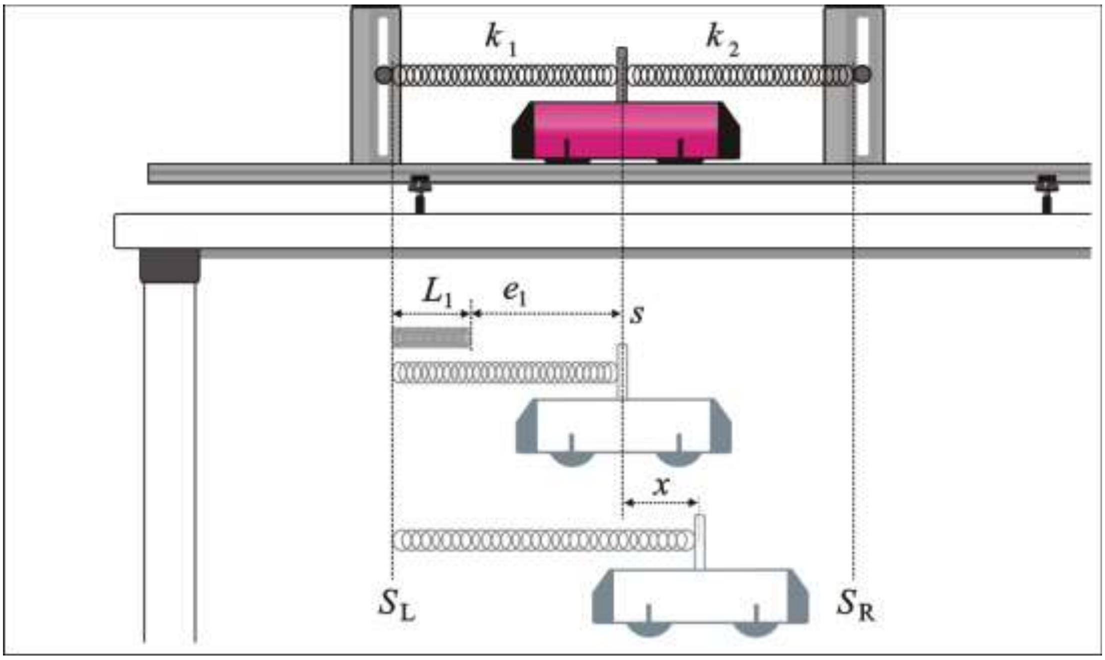
When RK is used to solve ODE and get the solution, apply the process of acquiring formula (5.2.7) and solve it.

[12] In Chapter 4 Collisions, this formula is expressed as the general form of RK and is used to explain the process of calculating the impulse in Excel.

5.2.3. Simulation Design

Simulation design begins with calculating the location and velocity using basic physical quantities and planning the control variables and initial conditions of the real experiments.

Let's consider the simulation of the system which is composed of two pulling springs and a cart. This situation is different from pushing-pulling spring situation, but the theoretical modeling process of calculating the general solution is the same. Only the variables which correspond to the initial conditions are different. The formulae of location and velocity can be applicable in the same way.



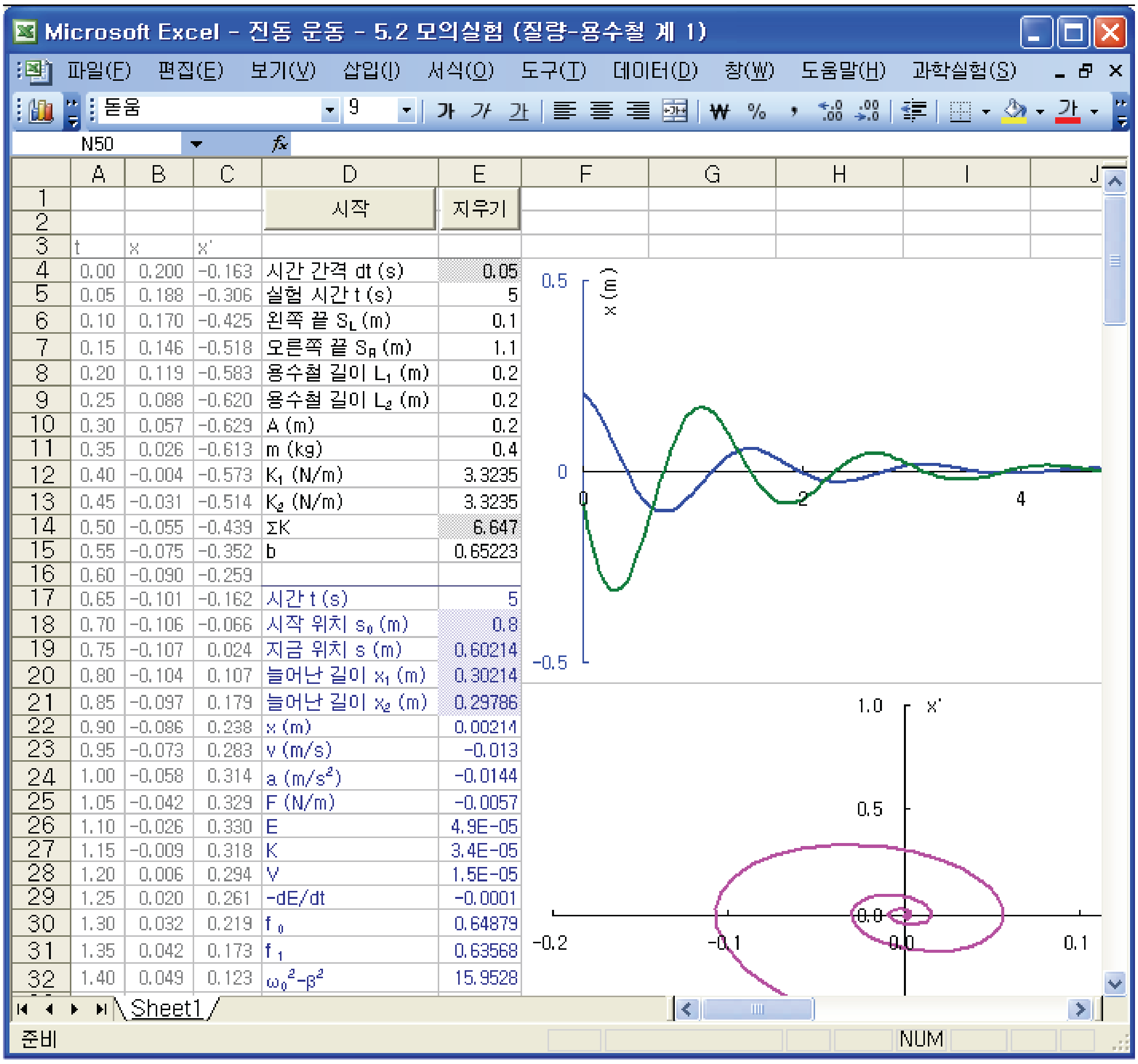
Picture 5.2.2 process of simulation design of a mass-spring system: When there is only one cart, the equations of motion can be made in case of slopes or spring pendulums and can be designed in the same way.

Picture 5.2.3 is a scene in which the simulation is done in Excel. The experiment like picture 5.2.1 was assumed, the initial conditions were designed, and the physical quantities from the general solution solving process were used in the simulation. The result values of time, location, velocity are recorded in row A, B and C of worksheet, and the initial values are recorded from E4 to E15. From E17 to E32, the result values per the time interval dt are recorded and the location, velocity and phase space are drawn as a chart in the scene.

In “Sheet 1” scene, the simulation is operated by clicking [Start] button. Therefore, $x-\dot{x}$ graph can be analyzed and compared to $x-t, \dot{x}-t$ graphs, whose initial conditions were various.

The designing process of simulations is the basis for making VBA original codes. Based on this, simulations can be realized even if the oscillations are different from picture 5.2.2.

This process is the basis for modeling physical developments with theories and experiments. Therefore, you should learn this to operate high level physical experiments.



Picture 5.2.3 simulation scene of mass-spring system: simulation can be operated by clicking [Start] button in worksheet.

Just like picture 5.2.3, if variables are designed as initial conditions to be input in the simulation scene, the result is as follows.

The modulus of elasticity of the system is the sum of each spring's modulus of elasticity and it should be calculated according to the experimental circumstances. In the circumstance of picture 5.2.2, the modulus of elasticity of the system can be written as $K = k_1 + k_2$. The value of K should be determined according to the experimental circumstances. The variables of initial conditions are in table 5.2.1 and 5.2.2.

S_L : The left end of spring 1
S_r : The right end of spring 2
L_1 : The length of spring 1
L_2 : The length of spring 2
A : Initial amplitude
m : Mass of the cart
k_1 : The modulus of elasticity of spring 1
k_2 : The modulus of elasticity of spring 2
K : The modulus of elasticity of the system consisted of spring 1, 2
b : Attenuation constant of the system

Table 5.2.1 the variables of initial conditions of the simulation[13]

In the track, the related variables to calculate the changes in the cart's location and the spring's length are as follows. These variables will be calculated within the Excel VBA program. These are the variables that will be used to calculate the location of the cart within Excel VBA program. Whether the location is wrong or not can be judged by the results of these variables. If the initial conditions are wrong when doing the simulation, the location of the cart will be wrong, too. So Excel VBA code should be made to stop the experiment.

[13] These variables are declared with the experimental circumstances such as picture 5.2.2. If the circumstances change, the designs of variables should be changed.

l_k : The ratio of length according to the modulus of elasticity
x_1 : The stretched length of spring 1
x_2 : The stretched length of spring 2
e_1 : The stretched length of spring 1 when it is in equilibrium
e_2 : The stretched length of spring 2 when it is in equilibrium
s : The location of the cart
x : The amplitude of the cart (displacement)

Table 5.2.2 internal variables which will be used in VBA code

Express the stretched lengths of spring x_1 and x_2 as the displacement x. The result is like below.

$$x_1 = e_1 + x$$
$$x_2 = e_2 - x$$

According to the equation of motion, $K = k_1 + k_2$ and $|k_1 e_1| = |k_2 e_2|$. Therefore, the system's total length L is as follows.

$$\begin{aligned} L &= S_R - S_L \\ &= L_1 + L_2 + e_1 + e_2 \end{aligned}$$

And,

$$L = L_1 + L_2 + e_1 + \left(\frac{k_1}{k_2}\right) e_1$$

$$\therefore\ e_1 = \left(\frac{k_2}{k_1 + k_2}\right) \cdot (L - L_1 - L_2)$$

If l_k is as follows,

$$l_k = \frac{L - L_1 - L_2}{k_1 + k_2}$$

e_1, e_2 are like below.

$$e_1 = k_2 \cdot l_k$$

$$e_2 = k_1 \cdot l_k$$

So the location of the cart s about the displacement x can be expressed like below.

$$s = (S_L + s_1) = S_L + (L_1 + x_1)$$

$$s = (S_R - s_2) = S_R - (L_2 + x_2)$$

$$\therefore s = \frac{1}{2}\left[(S_L + S_R) + (L_1 - L_2) + (x_1 - x_2)\right]$$

If the relationship between x_1, x_2, e_1, e_2 is used, the result is as follows.

$$(x_1 - x_2) = (e_1 - e_2) + 2x$$

$$= (k_2 - k_1)l_k + 2x$$

As a result, the location of the cart is like below.

$$s = \frac{1}{2}\left[(S_L + S_R) + (L_1 - L_2) + (k_2 - k_1)l_k + 2x\right]$$

The displacement of the cart x can be calculated by applying the x solved from the process of solving general solution or the secondary ordinary differential equation. The formulae of simulation above can be applied within Excel VBA code and the simulation scene such as picture 5.2.3 can be made.

This process of analyzing physical circumstances mathematically[14] is helpful to understand and apply the physical theories better because it helps students search the roles of each variable and study the physical theories concerning variables.

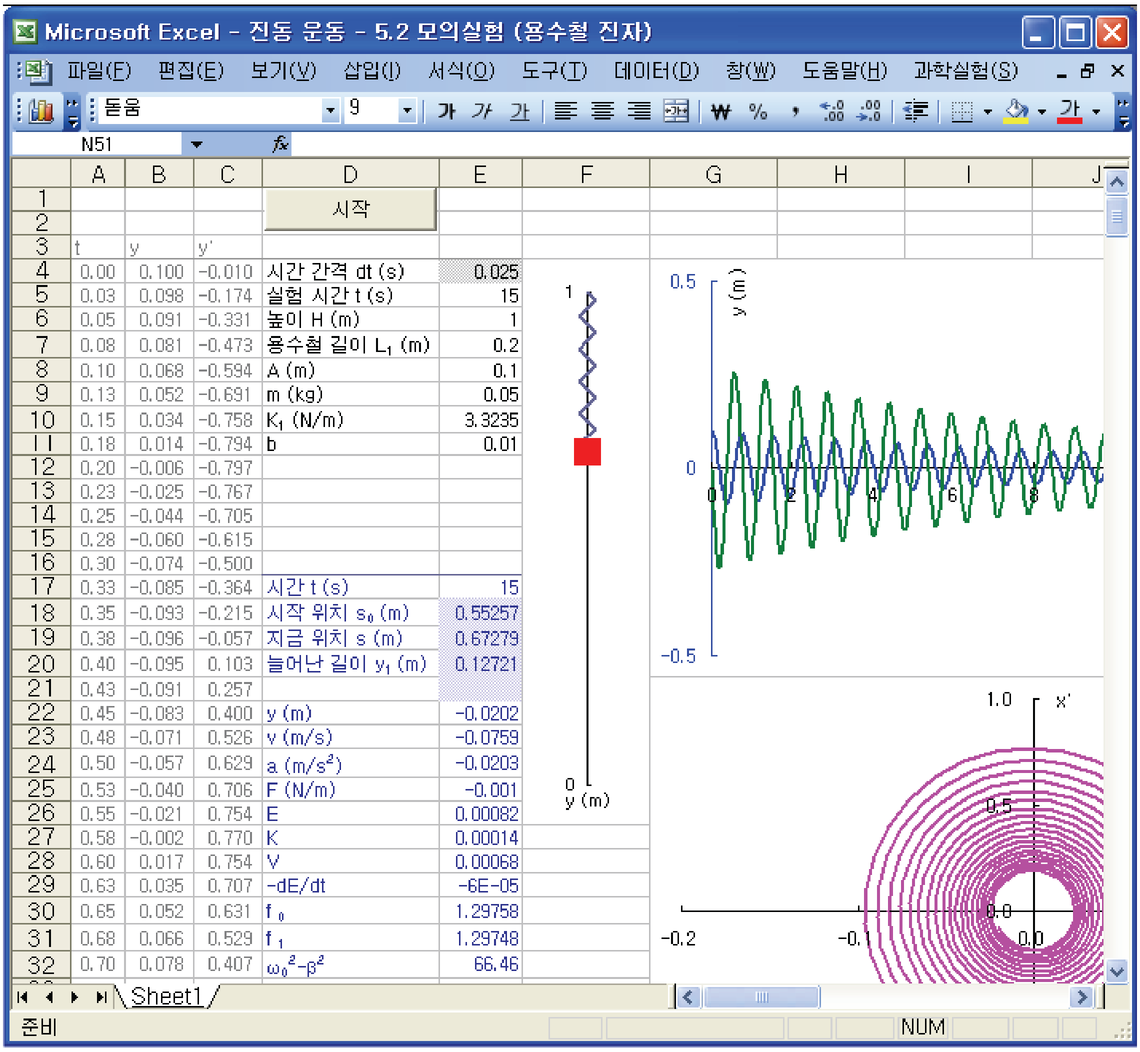
[14] This process is needed to the students who studies high level physical experiments in AP(Advanced Placement) process.

Exercise 5.2.3 Simulation design of a spring pendulum

Design the simulation of a spring pendulum which has periodic damping just like picture 5.2.1.

Explanation:

Picture 5.2.4 is a scene which includes the graph about the location, velocity and phase space of the spring pendulum.



Picture 5.2.4 simulation scene of a spring: the simulation will be operated by clicking [Start] button in “Sheet 1”.

The simulation scene explained by the general solution of the equation of motion is as picture 5.2.4. The location and velocity of the spring pendulum can be designed as a simulation by using the formulae of exercise 5.2.1 or 5.2.2. Exercise 5.2.1 gets the formulae of location and velocity by the solution of the equation of motion, so it can make Excel VBA code[15] more easily than RK of exercise 5.2.2.

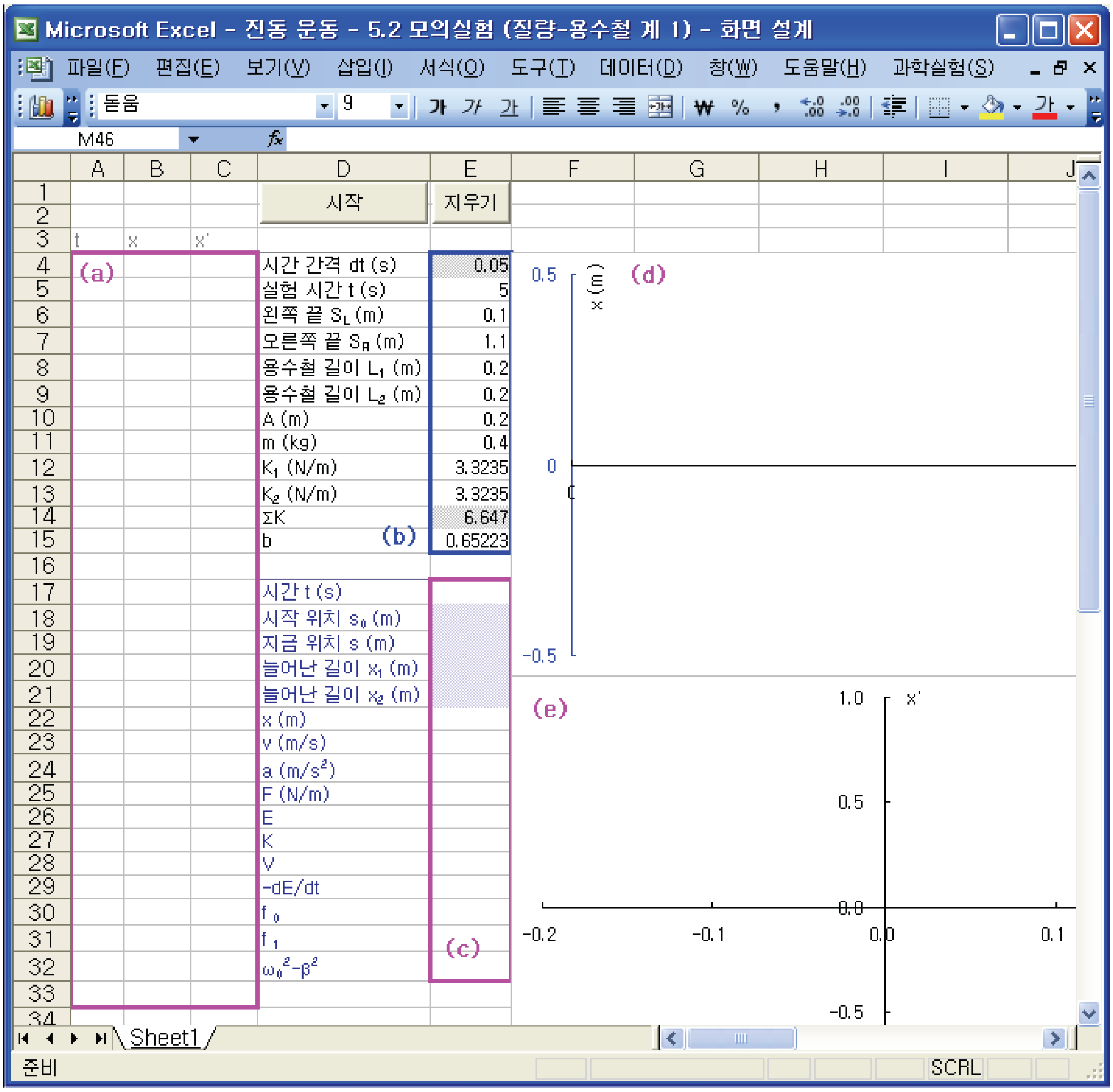
5.2.4 Simulation Making

We have learned the process of designing simulations. Based on these designs, let's learn how to realize simulations such as picture 5.2.2[16], which deals with the modeling of one-dimensional harmonic oscillator with Excel VBA program. According to the curriculum, this simulation making process can be omitted.

Picture 5.2.5 is the scene of simulation design. Set up the cell area for data recording as (a) in worksheet. (b) is the cell area for initial conditions, (d) and (e) are the cell areas for the results which will be calculated at the interval of dt, which was set up as initial condition. (e) is the chart that will show the graph which is drawn using the result data of (a). Graphs can express $x-t, \dot{x}-t, x-\dot{x}, E-t$ optionally. You can make graphs by using 차트마법사.

After designing the cell areas and making needed charts, you should make simulation start button by using [Order] button of [Control Tool Box].

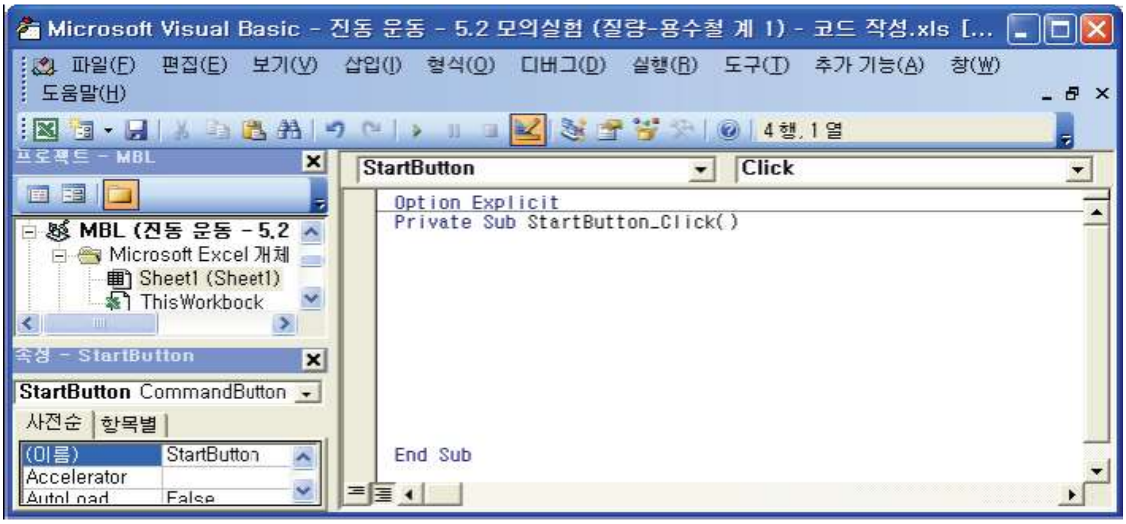
[15] More detailed information is in the Excel VBA original code, which is the supplement of this book.
[16] Once made, simulations can be used by anyone so that they can be applied to simulations of oscillations using Excel workbook files which store simulation sheets.



Picture 5.2.5 scene of simulation design about mass-spring system: In the scene, (a) is the cell for the result data of simulations, (b) is the cell for initial conditions, (c) is the cell for the result value of simulations, and (d) and (e) are the graph chart of simulations[17].

After finishing the scene designing, you should make program codes by selecting [Visual Basic Editor] of [Tool]. Picture 5.2.6 is the state of opening VBE window for the first time. If you set up the order button as Start Button, the program code that will be operated by clicking this button can be made in the sub procedure, that is, Private Sub StartButton_Click().

[17] The original data series range of the chart is the data in column A, B, and C.



Picture 5.2.6 VBE window for simulation code making: Project window, Property window and Code window (which can make programs) can be opened and the work will be operated.

Make VBA code within the sub procedure of Private Sub StartButton_Click(). First, make the declaration process[18] of initial condition variables and the variables which will be used within the program. Input initial conditions[19] to the cells of worksheet. Cells(RowIndex,ColumnIndex) has the property of substituting the values in the cells for the variables. Substitute the values of input initial conditions for the initial condition variables in the experimental circumstances and calculate the variables by the formula gotten from simulation designing. As the process of realizing the result graph of simulation, use For loop so that it can calculate at the interval [20]of dt the physical state[21] of a system. Set up the loop so that whenever the integer variable i increases one by one the time should flow as much as dt. And at the end of the For loop, the

[18] Initial condition variables and variables used in the program code are defined in the process of simulation designing. Declare variables to fit the regulations of VBA (Visual Basic Application) order system. Double is 64 beat numbers which show floating point and Integer is integer variables from -32,768 to 32,767.

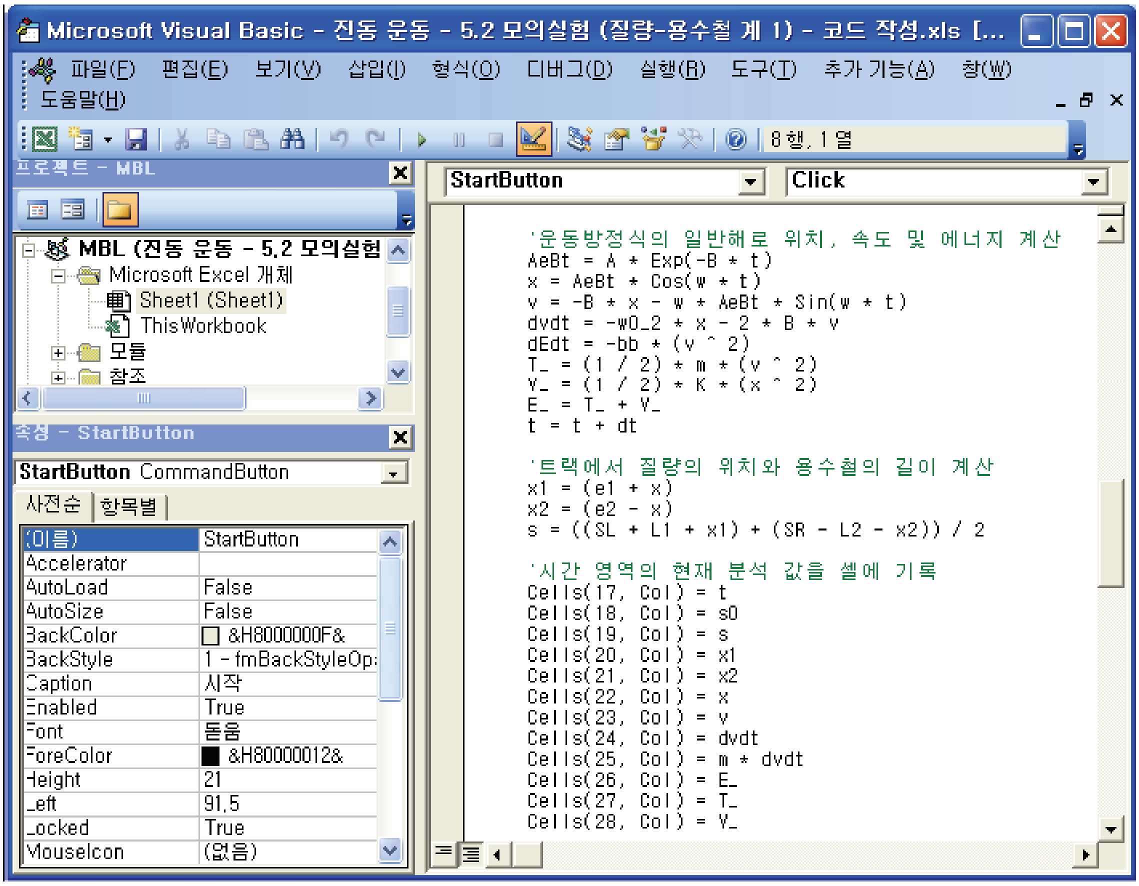
[19] You can use Cells property when inputting the initial conditions make within the cell area of worksheet. If the cell's location is changed in the scene design, modify the value of column and row in Cells property.

[20] In VBA program, the results of simulations will be shown at the worksheet of Excel at the interval of the established time. Here, it is the interval realized simulatedly in ODE, so it is different from the actual computer time. Because there are limits in the speed of VBA and limits in the exactitude of software-timed loop.

[21] Substitute the initial location and velocity of the cart and calculate the angular frequency of the system. Calculate the initial setting value to get the displacement and the stretched length of the spring and the location of the cart.

simulation should finish when it reaches the established time. Finally, calculate the location, velocity and energy by the formula of physical quantity which is calculated by the general solution of the equation of motion. Then calculate the location of the cart in the track[22]. The analysis results of the location, velocity and energy which will be used as the data of graphs should be recorded in the cell areas of worksheet.

Picture 5.2.7 is the part of the original codes which were made like this. After establishing VBA codes[23] like this, the simulation can begin by clicking the order button StartButton.



Picture 5.2.7 a part of VBA code making of simulations[24]

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[22] Calculate the value by the formula which fits the conditions of simulation. The formula of the simulation here is the formula of general solution which has periodic damped oscillation. Use this formula in For loop, which should be operated per the time interval dt.

[23] The established VBA code is stored automatically when storing Excel workbook. When opening the stored work book, the warning window asking macro security. So, VBA code can be operated after choosing Macro Including.

[24] The original VBA code can be downloaded and read at the site introduced in the supplement.

5.2.5 Simulation Conducting

Using simulations above, let's make charts concerning location, velocity, phase and energy by changing the initial conditions and check out the context using exercises. Input the initial conditions and attenuation constant and conduct simulations. Through the simulations, real experiment situations can be predicted and explained within the range of measurement error.

Exercise 5.2.4: Simulation of mass-spring system

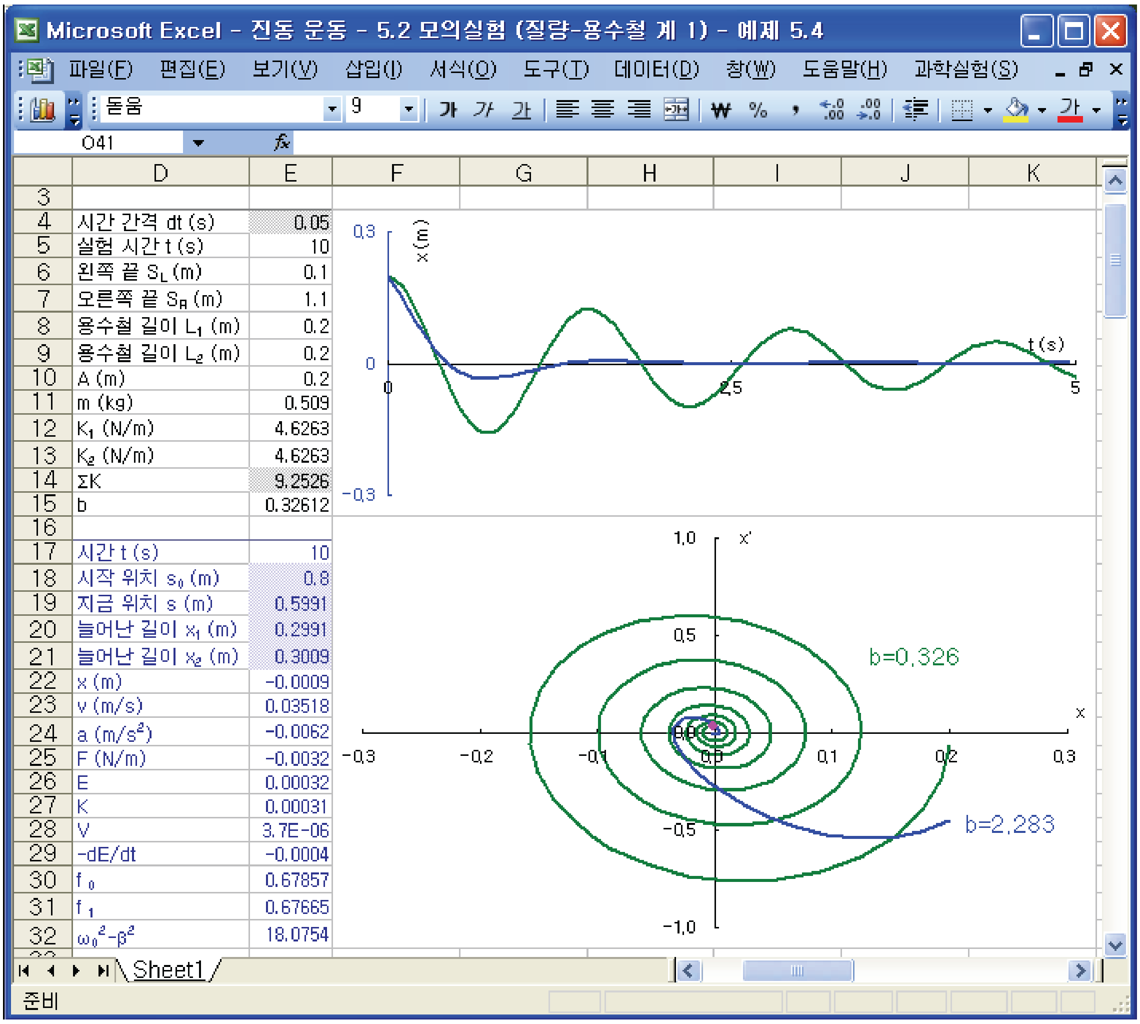
The graph about the phase and energy of strong damping can be calculated by using the simulation of mass-spring system.

<u>Solution:</u>

Picture 5.2.8 is the result of the simulation when the attenuation constants are b=0.326, b=2.283. Below is the initial conditions used in the simulation. Simulations can be conducted by changing these conditions.

Left end S_L	0.1
Right end S_R	1.1
Spring length L_1	0.2
Spring length L_2	0.2
A	0.2
m	0.509
K_1	4.6263
K_2	4.6263

Table 5.2.3 Initial conditions used in the simulation



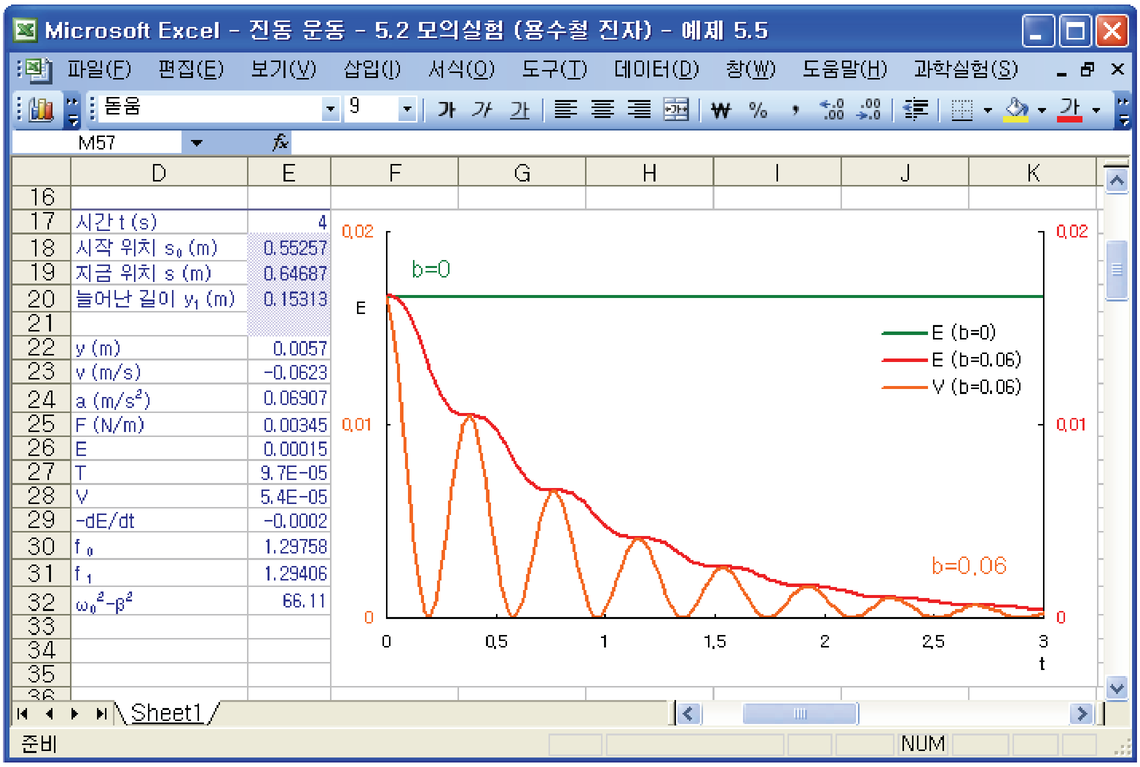
Picture 5.2.8 phase spaces of strong damping and weak damping

Like this, by changing the attenuation constants, the graph of strong damping can be made. The attenuation constant can be different according to the situations.

Exercise 5.2.5: The energy graph of a damped oscillation system

Conduct the simulation of spring pendulum and make (1) the graph of potential energy and total energy, (2) the potential graph of the system, (3) dE/dt graph, in case of no damping and weak damping.

Picture 5.2.9 is the graph of potential energy and total energy according to time. At the peak of potential energy, the curve of total energy meets it. That is, it shows that total energy of the system is the same as the maximum of potential energy.



Picture 5.2.9 Graph of potential energy (V) and total energy (E): Green E curve shows when there is no damping, Red E curve is when there is damping(b=0.06). Orange V curve is the potential energy graph when there is damping.

Picture 5.2.10 is the scene which adds dE/dt graph in the chart and expand[25] the time interval of picture 5.2.9 from 0 to 0.8 sec. Point a,b,c are when dE/dt, T, V are at the valley, and point d, e, f are when dE/dt, V, E, T are at the peak. At the valley, there are phase differences between a, b, and c. dE/dt, T and V shows the phase differences because the loss of energy can be calculated by the function of velocity when the damping occurs. Point d, e, f are all the same time, so E is at the maximum when V is at the maximum, and T and dE/dt are zero.

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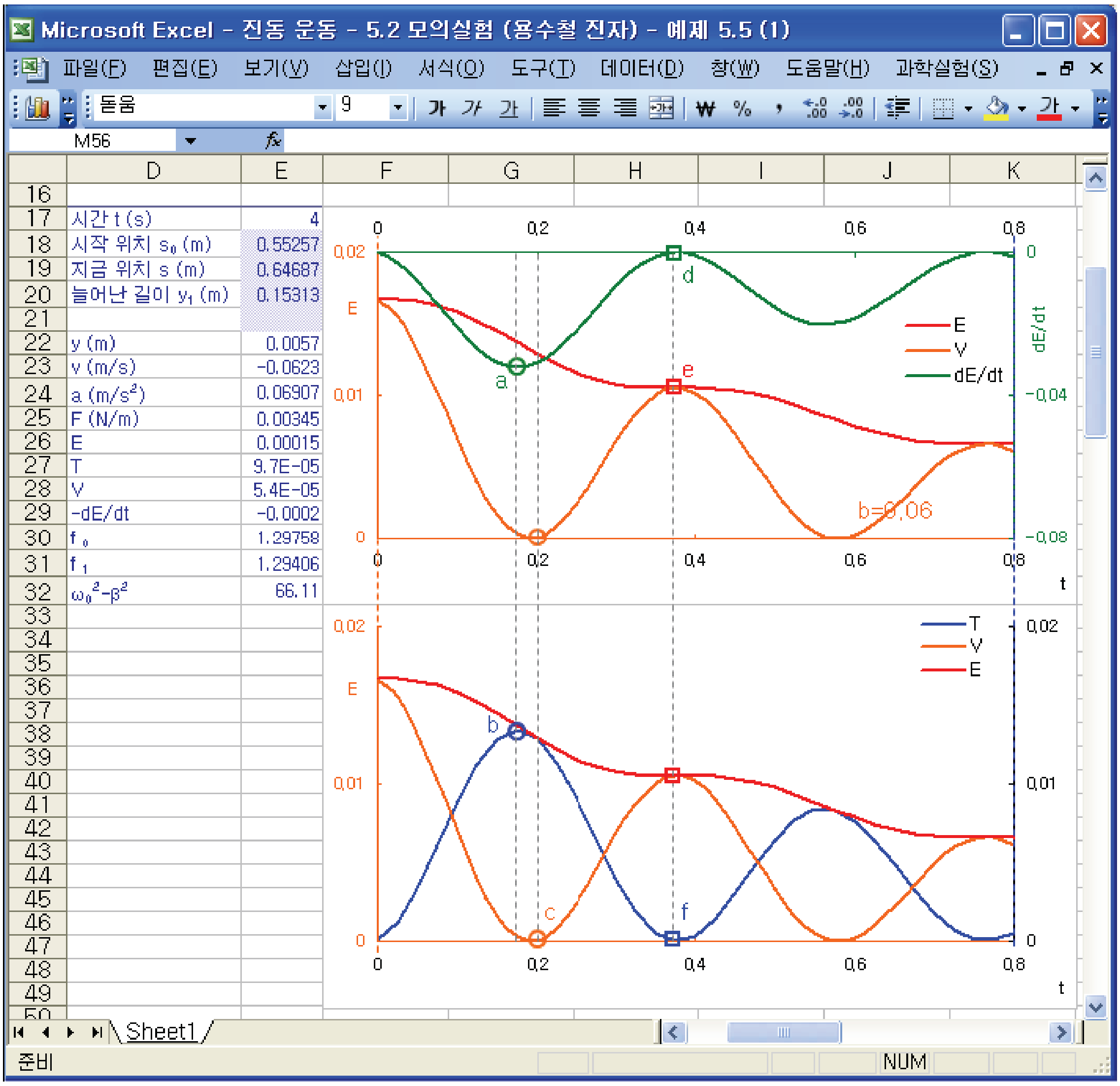
[25] To enlarge and reduce the graph's scales in Excel, change the minimum and maximum of [X Scale] in [Axis Form]-[Scale] of the chart.

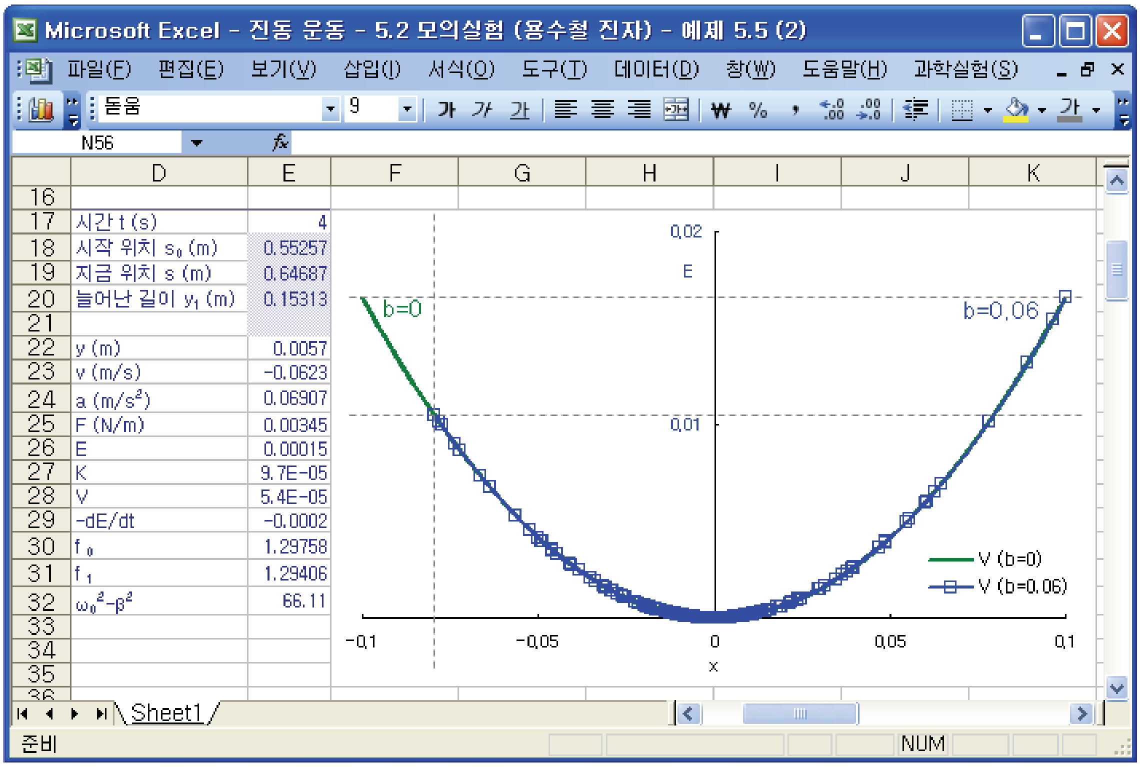
In the state of oscillation, when there is damping, the energy generally changes as time flows. This is shown in the phase space graph of location and velocity.



Picture 5.2.10 graph of the system's energy and dE/dt: a is the valley of dE/dt graph, b is the peak of kinetic energy, c is the valley of potential energy, d is the peak of dE/dt, e is the peak of potential energy, and f is the valley of kinetic energy.

When there is no damping, the peaks of potential energy in picture 5.2.10 have same heights. In picture 5.2.11, the curve of potential energy according to the cart's location shows the potential of a harmonic oscillator which shows bilateral symmetry on $x=0$. When there is damping, the potential decreases every time interval dt and later it becomes 0.

In the graph of picture 5.2.10, total energy curve passes the peak of potential energy, and there are phase differences between the valley of potential energy and the peak of kinetic energy. This is the result of the simulation which shows the conversion and loss of energy in the process of oscillation.



Picture 5.2.11 graph of the system's potential energy and time V-t

Exercise 5.2.6 Calculating the attenuation constant Q

With reference to the simulation design and codes of picture 5.2.3, conduct the simulation which is about the motion of mass-spring system which has periodic damping, and calculate the graph of the modified phase x and $\dot{x}+\beta x$ and the attenuation constant Q(quality factor).

Solution:

The result of simulation is as below. This simulation is conducted with the modulus of elasticity 22.895N/m, the attenuation constant 0.05913, the cart's mass 0.5328kg and the length of the spring 0.142m.

In the oscillation that has weak damping, the energy loss of the system can be characterized as Q. Q can be calculated as below.

$$Q = \omega_1 \tau = \frac{\omega_1}{2\beta}, \quad \tau = \frac{1}{2\beta}$$

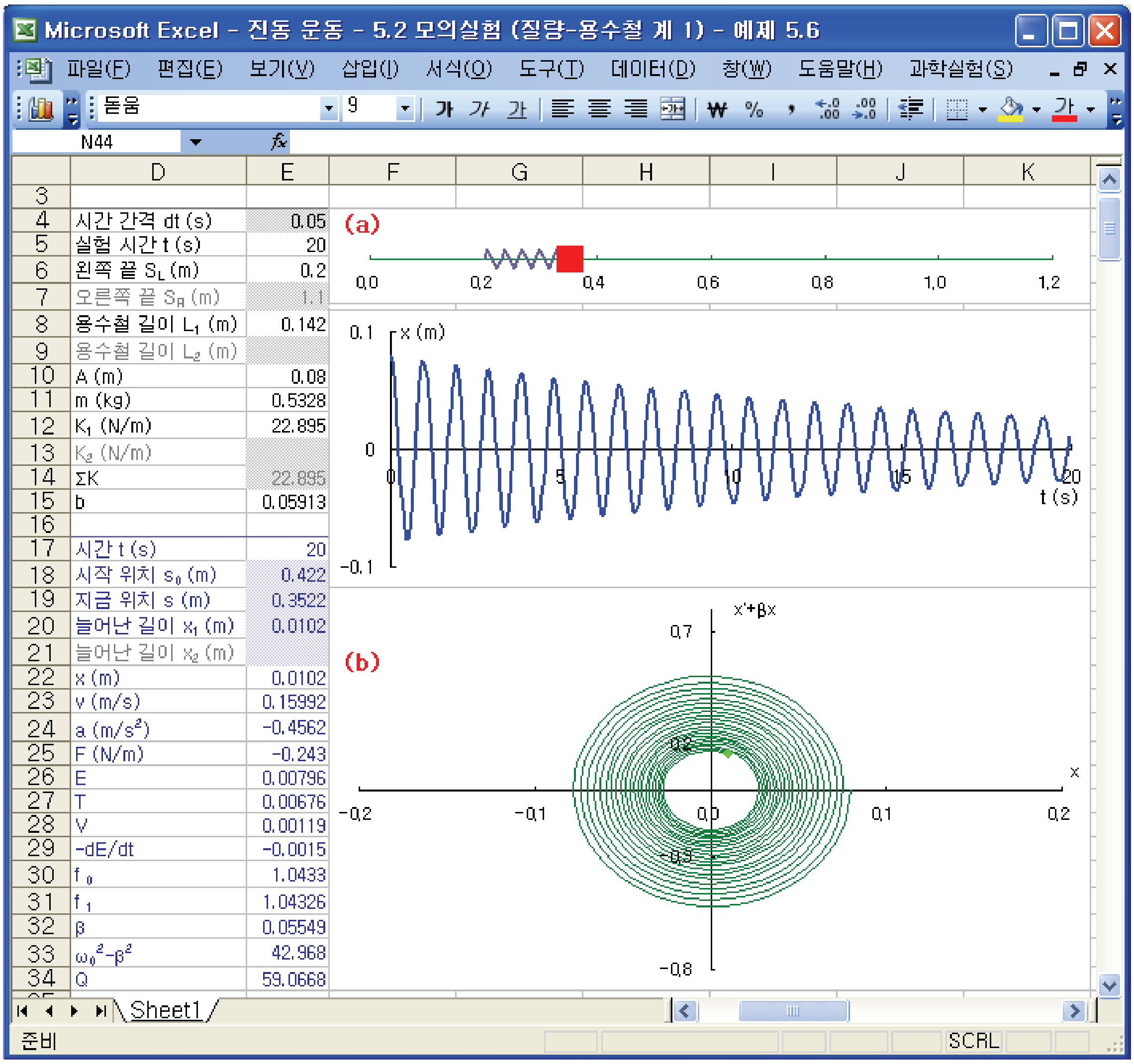
The result of simulation analyzing is as follows. This result can be compared to the real experiment result. In the simulation, you can learn that $\omega_0 \approx \omega_1$.

f_0	1.043298
f_1	1.043260
β	0.055487
$w_0^2 - \beta^2$	42.968017
Q	59.066832

Table 5.2.4 the result of simulation: natural frequency f_0, damped frequency f_1, attenuation constant β

In picture 5.1.12, (a) shows the motion of the spring[26] during the simulation. (b) is the graph of the phase space which is modified[27] as x and $\dot{x} + \beta x$. The result graph of the simulation can be compared with the graph of real experiment because it is the physical model of theoretical prediction.

[26] This is a computer simulation for the understanding of the oscillations.
[27] Use x and $\dot{x} + \beta x$ graph instead of $x - \dot{x}$ graph.



Picture 5.2.12 Simulation of periodic weak damping[28]: b=0.05913, m=0.5318, K=22.895

In 5.3 Experiment Analysis, the data analyzing and the result interpretation of the oscillation will be introduced. The simulations and experiment analyses in chapter 5 can be used as tailored educational materials according to the level of the students. The mathematical methods and VBA codes used in the simulations will be helpful when learning mathematical physics.

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[28] Simulations can be conducted by changing b value, which is the condition of damping. The modulus of elasticity K is different according to the type and character of the spring, so the K value of spring used in the real experiment should be used.

5.3. Experiment Analysis

In the experiments, unlike simulations, the initial conditions will be less. The variables that should not be input are such as the length of the track or the spring. These variables are needed in the simulation to calculate the cart's location, but it can be calculated directly by the motion sensor. Also, when the spring's modulus of elasticity or the mass of the system is unknown, the damped constant β, the angular frequency ω_0 and ω_1, the Q value and the custom formula about the system's motion can be calculated using the measured data of the cart's location.

The theoretical equation of motion used in the experiment analysis is like formula (5.1.8). The real experimental data can be expressed like formula (5.3.1), which changes the curve of amplitude into the general form which has the intercept.

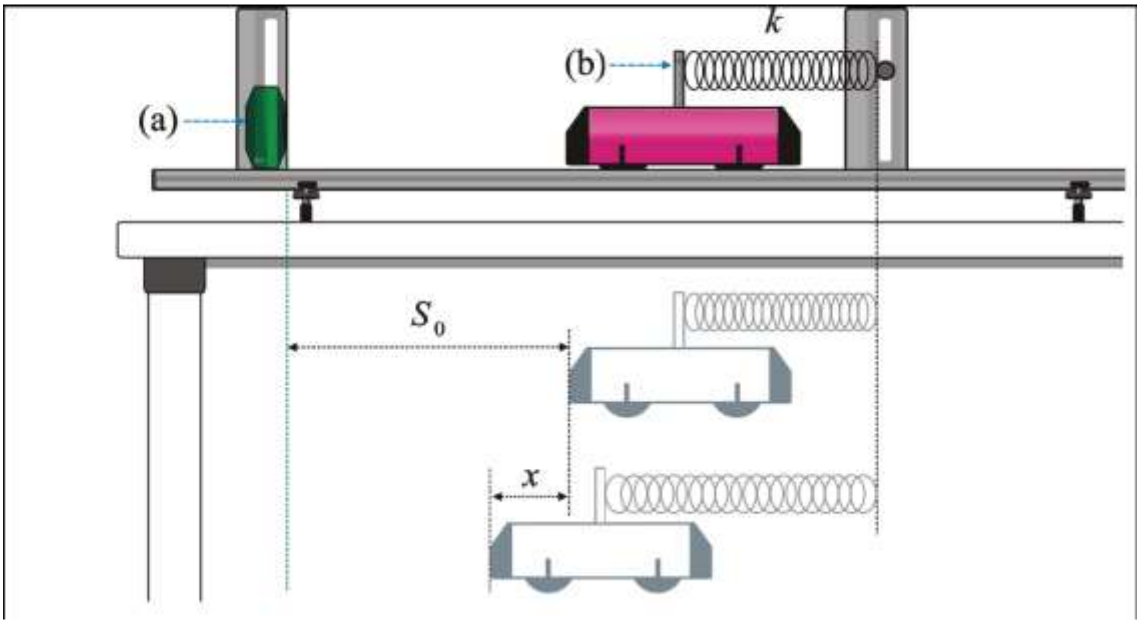
$$x_A = Ae^{-\beta t} + C \qquad (5.3.1)$$

$$x = x_A \cos(\omega_1 t) = [Ae^{-\beta t} + C] \cdot \cos(\omega_1 t) \qquad (5.3.2)$$

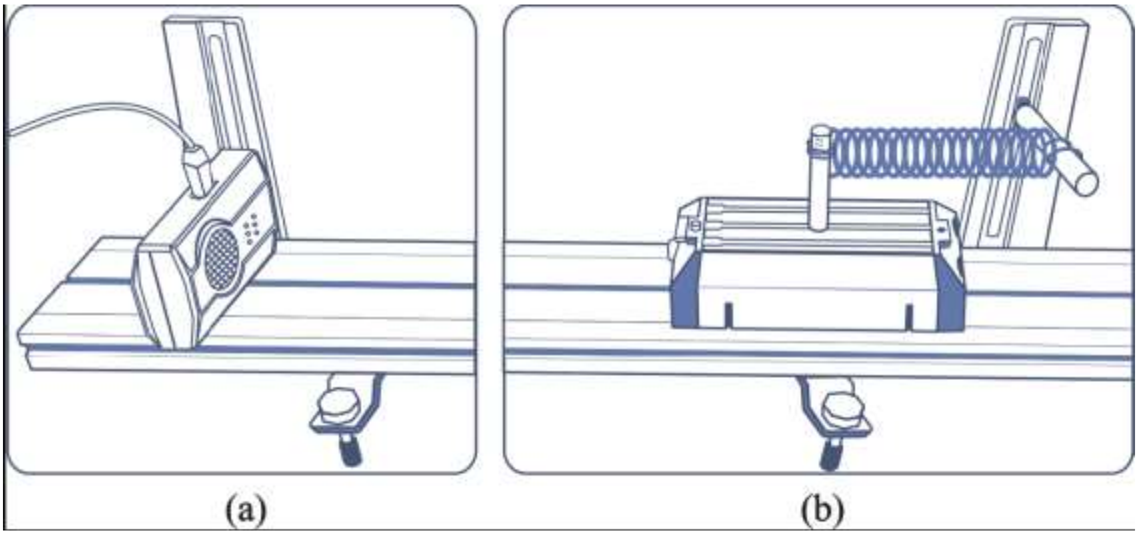
When the measured data of the oscillation is acquired, the physical value about the system's motion can be calculated and the motion can be explained by analyzing the curve of amplitude (which is the envelope) and the frequency in the graph of displacement and time. The analysis of frequency, which is like formula (5.1.8), is to calculate the angular frequency ω in the formula of the damping, and the curve of amplitude is to execute the exponential curve fitting in the form of the equation of motion, just like formula (5.1.8). The process to conduct these two is as follows. First, let's check out the experimental circumstance of the oscillation.

5.3.1. Experimental Circumstance

Picture 5.3.1 is the experimental circumstance of the oscillation using a pushing-pulling spring and a cart. The setting of the motion sensor (a) and the spring (b) should be like picture 5.3.2. The experiment can be designed differently by changing the installation of the motion sensor so that it can sense the location of the cart according to the circumstance of the oscillation such as the motion in the slope, motion using a pulley, motion of a system using two springs and a spring pendulum.



Picture 5.3.1 experiment of a mass-spring system: Estimate the distance which the cart moved by a motion sensor. The measured data is the location of the cart on the track.

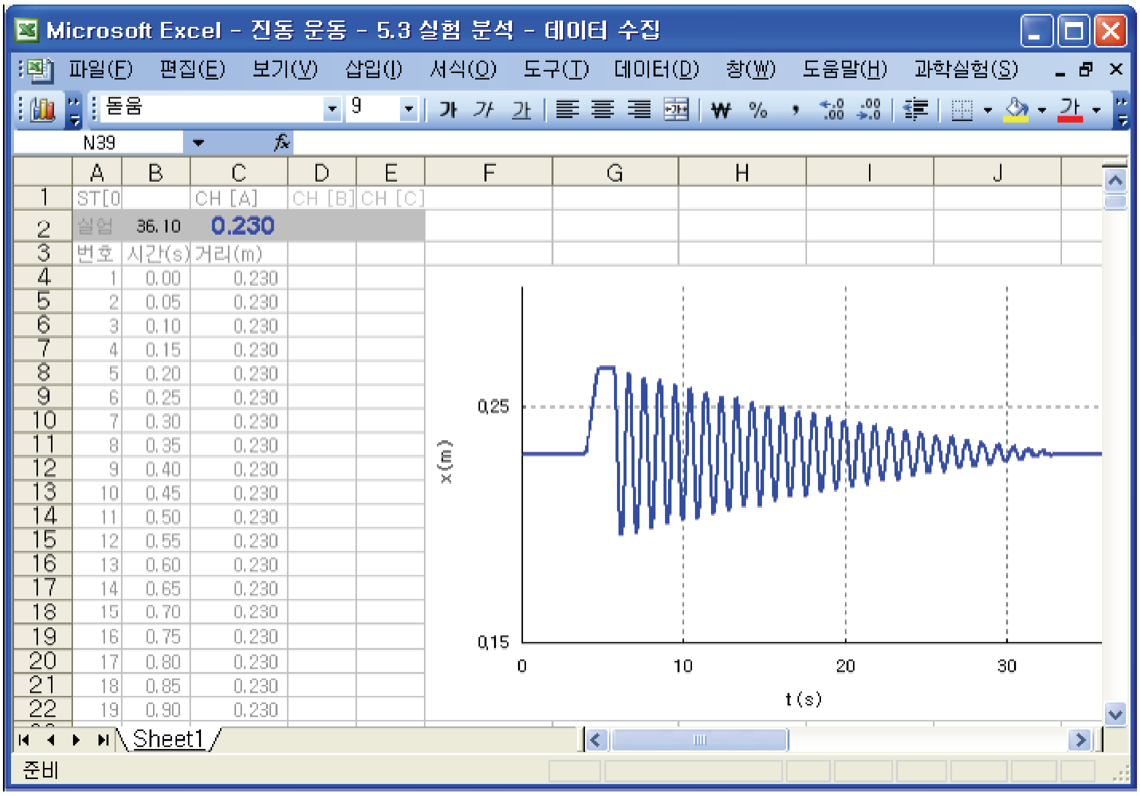


Picture 5.3.2 the way of installing the motion sensor and the spring: Like picture (a), the motion sensor should be combined with the sensor bracket. The spring should be sent up above the track so that it should not be dragged by reaching the track[29].

Picture 5.3.3 is the scene of opening the worksheet “Sheet1” of Excel, which collected the data in this experimental circumstance. When the motions sensor is connected to channel A and data is collected in worksheet “Sheet1” of Excel, the time is recorded in column B, and the location is recorded in column C. Many sheets can be added in Excel workbook, so the experiment can be done repeatedly in the same experimental circumstances and can be stored as one workbook file.

Although the circumstances are different, the experimental data can be collected in the worksheet in this way, so the way of analyzing the amplitude and frequency can be applied identically regardless of the experimental circumstances and the initial conditions. For example, in case of different circumstances such as the oscillation of a spring pendulum or the oscillation of a cart on the one dimensional track, the data analyzing of the oscillation is identical.

[29] If the spring is dragged, the irregular damping will occur with the friction of the track and the periodic damped oscillation cannot be maintained. As in picture 5.3.2, the instrument such as the support can be used.



Picture 5.3.3 the result of collecting data concerning the oscillation of a cart in Excel workbook[30]

As in picture 5.3.3, the scene design for experiment analysis using "Sheet 1" should be drawn up in a new sheet named "Analysis", like picture 5.3.4. From now, let's find out the way of drawing up the analysis sheet. According to the level of the curriculum, this process can be omitted[31]. The students in the high physical experimental level can improve their ability of experiment analysis by conducting this process on their own.

The basic physical value to be analyzed is as table 5.3.1 below. Time, location, velocity and energy are the physical values recorded in the cells of the worksheet per the time interval dt . This analysis sheet can be used in various experimental circumstances to analyze the experimental data.

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[30] Regardless of the experimental circumstance, the data of time and location is collected in column B and column C of worksheet "Sheet1".

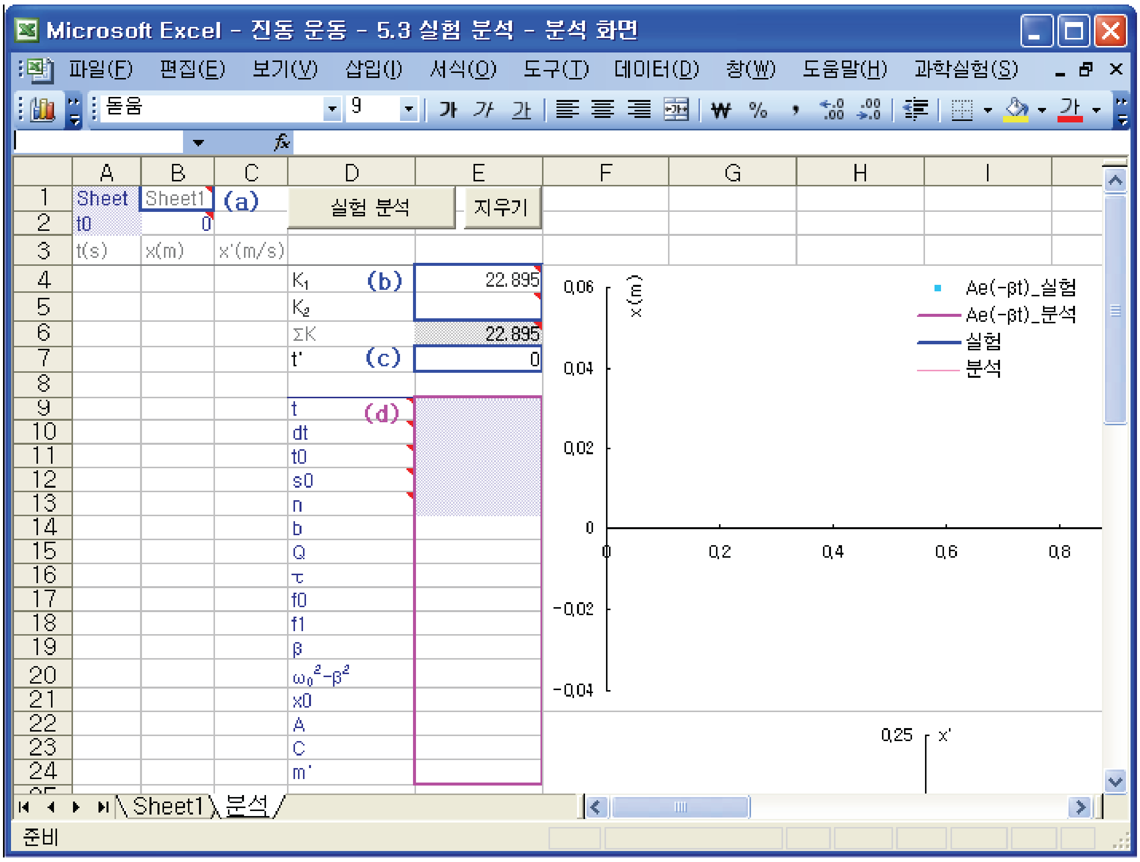
[31] Once drawn up, the analysis sheet can be used to analyze the experiments of oscillation. This sheet does not need to be drawn up by every student. Distribute to the students before the experiment.

Time t'	the later time of the time section
Time t	time of experiment completion
Time dt	estimating interval
Time t_0	the first time of the time section
Location S_0	location indicating the system's equilibrium: the oscillation center of x
Peak n	the number of the peak
Constant b	damping resistance
Constant Q	Q-constant
Constant r	time constant
Frequency f_0	natural frequency
Frequency f_1	damped frequency
Constant β	damping constant
Value $\omega_0^2 - \beta^2$	damping condition
Displacement x_0	amplitude of the system
Constant A	modulus of the amplitude
Constant C	intercept of the amplitude
Mass m'	mass of the oscillating system

Table 5.3.1 Physical values to be analyzed in the experimental circumstance of oscillation

In table 5.3.1, time t' is the later value of the time section which should be analyzed[32]. In picture 5.3.4, the name of the sheet to be analyzed should be input in (a) cell B1, the modulus of elasticity in (b) cell E4 and E5, and time t' in (c) cell E7. For instance, when the total experimental time is 50 seconds, if you want to analyze from 0 to 35 seconds only, put number 35 into cell E7. If t' is 0, data can be analyzed from the first till the end of the time. In (b), when the modulus of elasticity K_1 or K_2 is input, the total modulus of elasticity $\sum K$ should be calculated in cell E6.

[32] In the real experiment, unlike the simulation, there may be the section in which the damping constant itself changes according to the system's circumstances. For example, when the damping occurs enormously in the air, the damping of the amplitude can be different according to the sections.



Picture 5.3.4 the worksheet in which the experimental data of oscillation will be analyzed[33]

5.3.2. Analyzing Process – Damped Oscillation

Let's draw up the analysis sheet. The analysis sheet calculates the peak of the amplitude curve from the location data expressed in time and distance and then calculates the custom modulus of the exponential curve of the amplitude. From the value of the peak, the frequency can be calculated and the damped angular frequency ω_0 and ω_1 can be calculated, too. Using the modulus of the amplitude curve, the curve of the experimental data and the curve of the analysis result can be compared. Next is the summary of the experimental data's analysis process in the analysis sheet.

[33] If you write the sheet's name in cell B1 and click [Experiment Analysis], the analysis will be done automatically and the result will be recorded in the cells.

a. Calculate the peak of the amplitude curve $P_i(x,t)$.

b. Calculate the location indicating the system's equilibrium from valid peaks.

c. From the values of peaks, calculate the damped frequency f_1.

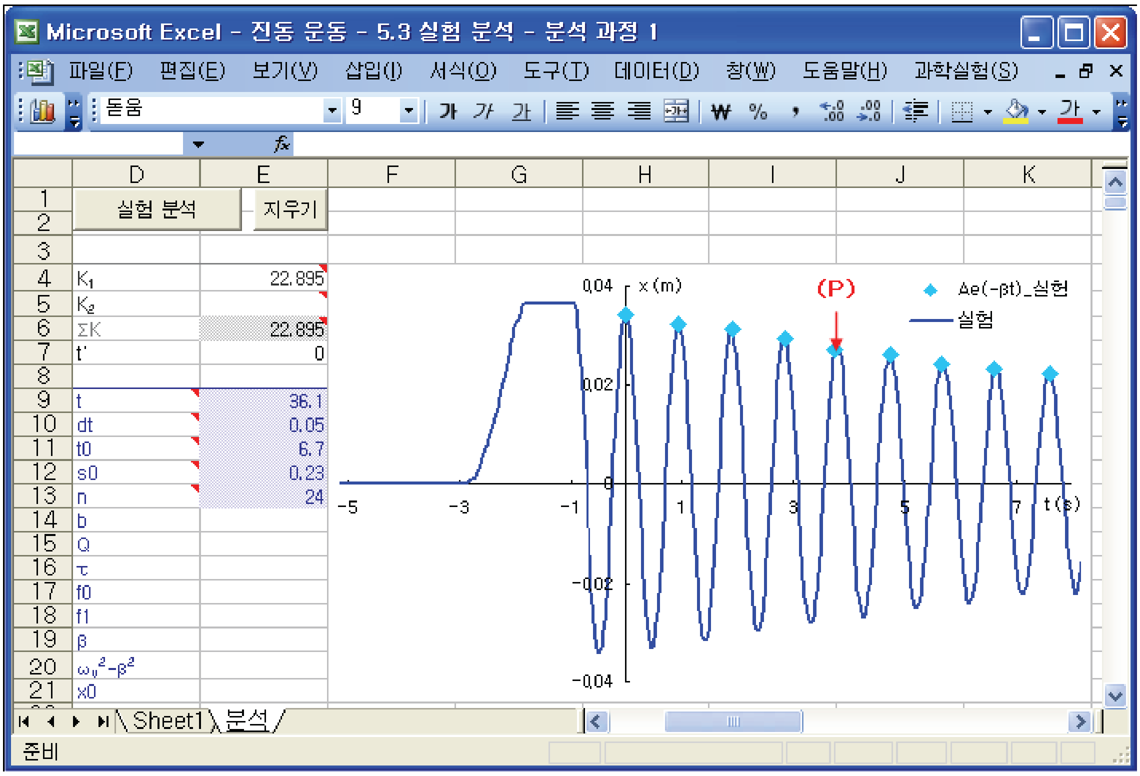
d. Execute the exponential curve fitting of formula 5.3.1.

e. Calculate the damping constant β, the moduli of the curve A and C.

f. Execute the fine tuning to the moduli A and β.

g. Calculated various physical values about the system's motions.

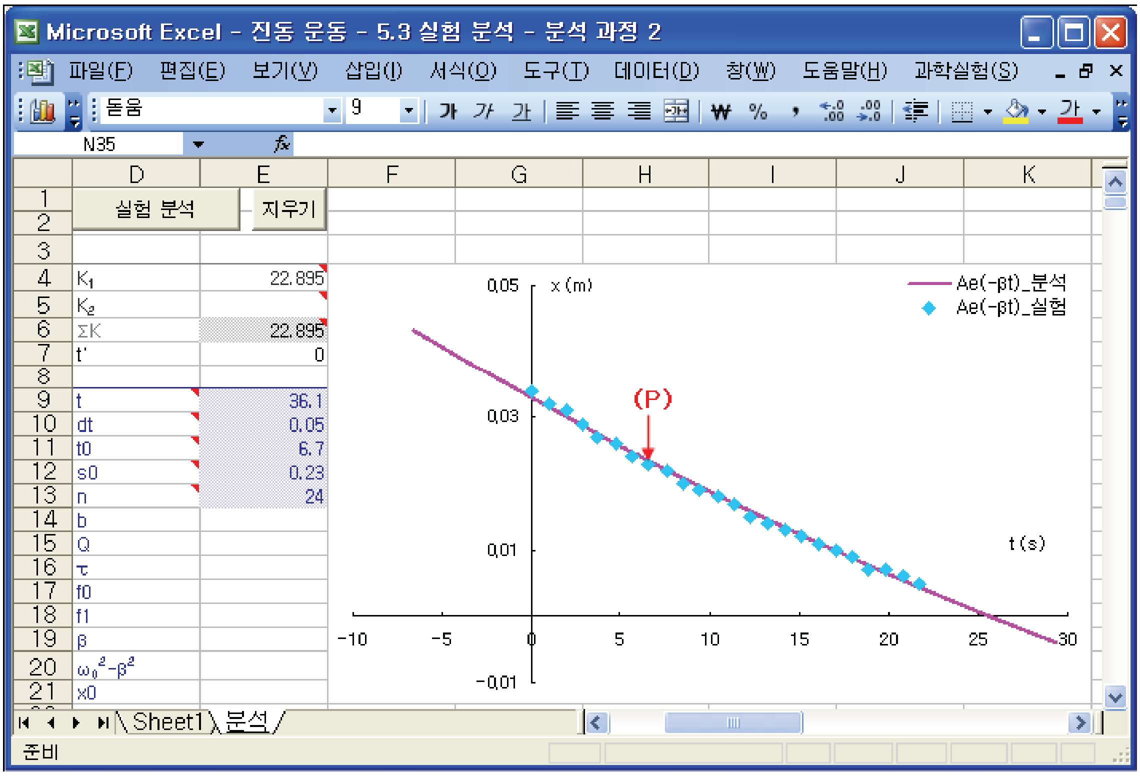
First, calculate the arrangement $P_i(x,t)$ of the peak of the amplitude curve which is the envelope of the motion graph. Picture 5.3.5 shows that the peak values are expressed in points (P).



Picture 5.3.5 Analyzing process 1 – Calculating the peak of the amplitude curve: Analyze the value of the peaks and calculate the damped frequency of the system.

Frequency f_1 is calculated by acquiring the average value of time t_i in the arrangement of the peak values and using the angular frequency $\omega_1 = 2\pi f_1$. When calculating $P_i(x,t)$ mathematically, you should judge whether x_{i-1}, x_i, x_{i+1} increases or decreases when the widths of the peak observation are three before and after the value of P_i . If you set up the observation widths as five levels, observe the values of $x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}$.

Second, arrangement $P_i(x,t)$ becomes the points which pass the formula of the amplitude curve (5.3.1), so the moduli A, C and the damping constant β can be calculated by the exponential curve fitting. The result of the exponential curve fitting is as picture 5.3.6.



Picture 5.3.6 analyzing process 2 – Exponential curve fitting: From the arrangement $P_i(x,t)$ which shows the peak values of the amplitude curve, calculate the moduli A, C and the damping constant β . The solid line in the graph is the formula of the exponential fitting curve $Ae^{-\beta t} + C$.

The modulus of formula (5.3.1) should be done fine tuning using GROWTH and LINEST functions of Excel[34]. By the fine tuning, the formula of the exponential curve fitting can be calculated more accurately within the error range. If the displacement of

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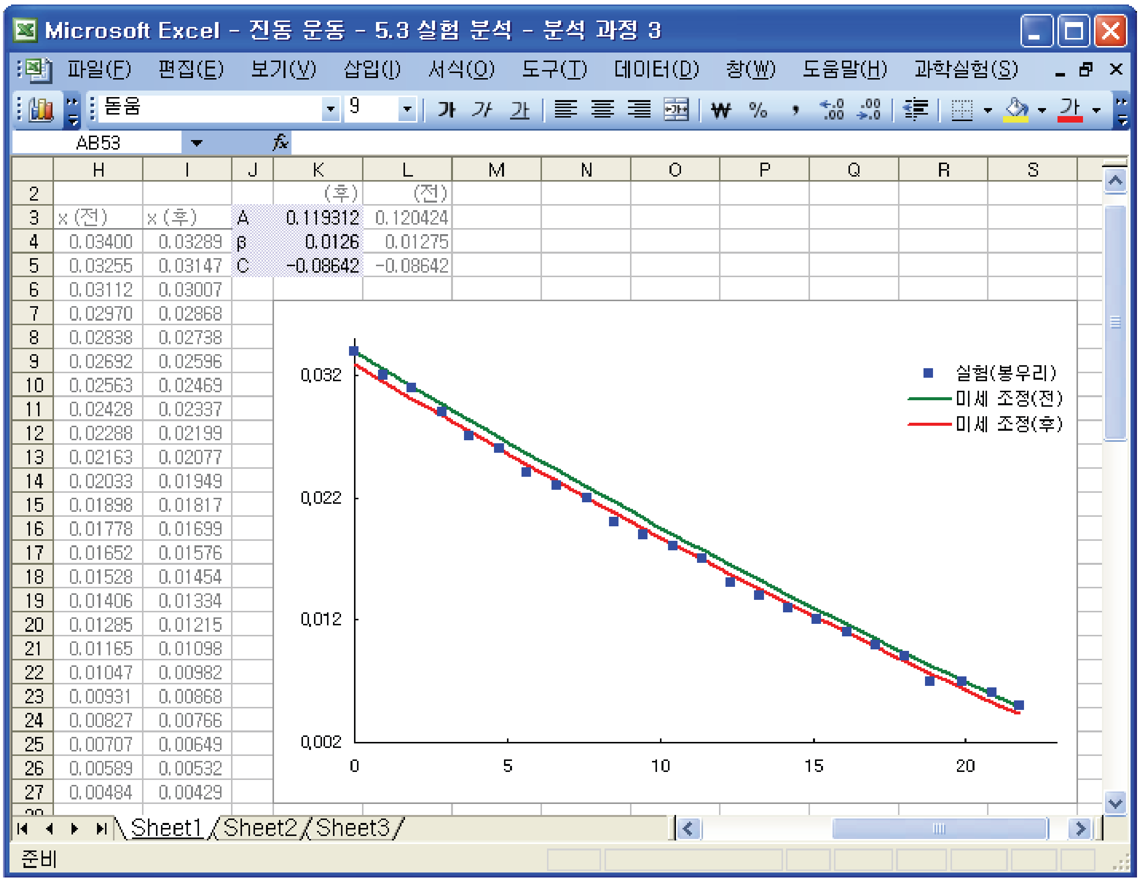
[34] The formula of the amplitude curve can be executed the simple exponential curve fitting by GROWTH and LINEST functions only. In 5.3.2, there is the analyzing process of the exponential curve fitting, which is the process of calculating the modulus of the exponential curve directly before using GROWTH and LINEST functions. This explains the mathematical analyzing process of physical experiments.

the amplitude swings a lot within the error range, the importance of the fine tuning should be emphasized for accurate experiment analysis. Picture 5.3.7 shows the results of the case with fine tuning and without fine tuning on the graph.



Picture 5.3.7 analyzing process 3 – fine tuning: This is the result of fine tuning for A and β, which are the moduli of the exponential fitting curve that passes $P_i(x,t)$. $P_i(x,t)$ is shown as points in the graph and it is the arrangement of the peak values of the amplitude curve.

After the fine tuning, the formula made by the exponential curve fitting[35] and the angular frequency ω can be calculated. Then, using formula (5.3.2), the damped oscillation curve from the experimental data and the curve from the analysis result can be compared with picture 5.3.7. The mathematical process of the exponential curve fitting is as follows. When the exponential curve fitting is applied like formula (5.3.1), if $B=-\beta$, the amplitude's displacement x and velocity $\dot{x}$ is like below.

---

35 About the exponential function $x = Ae^{\beta t} + C$, the moduli A, B and C can be calculated directly by GROWTH function of Excel. Solve ln(GROWTH(x−C))=Bt+ln(A), C= | X_A− GROWTH(| x_A |) | .

$$x = Ae^{-\beta t} + C = Ae^{Bt} + C$$

$$\dot{x} = ABe^{Bt}$$

B can be calculated first from the formula of $\dot{x}$ which does not have constant C. If B_i is calculated from $\dot{x}_j$ and $\dot{x}_i$, the result is as follows.

$$\frac{\dot{x}_j}{\dot{x}_i} = \frac{ABe^{Bt_j}}{ABe^{Bt_i}}$$

$$\ln\left(\frac{\dot{x}_j}{\dot{x}_i}\right) = B(t_j - t_i)$$

$$B_i = \frac{\ln\left(\frac{\dot{x}_j}{\dot{x}_i}\right)}{t_j - t_i} \qquad i = 0,\ 1,\ 2, \ldots \qquad (5.3.3)$$

In formula (5.3.3), when B is $x_j = x_{i+1}$ and $t_j = t_{i+1}$, the result is like below.

$$\frac{\dot{x}_j}{\dot{x}_i} = \frac{\dot{x}_{i+1}}{\dot{x}_i} = \frac{\left(\frac{x_{i+2} - x_{i+1}}{t_{i+2} - t_{i+1}}\right)}{\left(\frac{x_{i+1} - x_i}{t_{i+1} - t_i}\right)}$$

$$t_j - t_i = t_{i+1} - t_i = \left(\frac{t_{i+2} + t_{i+1}}{2}\right) - \left(\frac{t_{i+1} + t_i}{2}\right)$$

$$= \left(\frac{t_{i+2} - t_i}{2}\right)$$

And i is the number of the peak values. Constant B can be acquired by calculating the average of B_i values from formula (5.3.3).

Let's assume constant A. Using x_{i+1} and x_i, the result is like below.

$$x_i = Ae^{Bt_i} + C$$

$$x_{i+1} = Ae^{Bt_{i+1}} + C$$

$$x_{i+1} - x_i = A\left(e^{Bt_{i+1}} - e^{Bt_i}\right)$$

$$A_i = \frac{x_{i+1} - x_i}{e^{Bt_{i+1}} - e^{Bt_i}} \qquad (5.3.4)$$

The arrangement value A_i can be calculated like formula (5.3.4) and the average of this value indicates constant A. After this, let's assume constant C.

$$x_i = Ae^{Bt_i} + C$$

$$C = x_i - Ae^{Bt_i}$$

So, when the time $t_0 = 0$, C becomes like below.

$$C = x_0 - Ae^{Bt_0} = x_0 - A \qquad (5.3.5)$$

Lastly, let's execute the fine tuning to constant A and B. When the exponent growth prediction curve which passes $P_i(x_i, t_i)$ is expressed using GROWTH function of Excel, the result is as follows.

$$\mathrm{GROWTH}(x_i - C) = A' e^{B' t_i}$$

$$\ln\left(\frac{\mathrm{GROWTH}(\mathrm{x_i} - \mathrm{C})}{A'}\right) = B' t_i \qquad (5.3.6)$$

So, if formula (5.3.6) is expressed like formula (5.3.7), which is the simple function in the form of $Y = bX + a$ and the gradient and intercept are calculated by LINEST function of Excel, new fine tuned A' and B' can be acquired.

$$\ln\left[\mathrm{GROWTH}(x_i - C)\right] = \ln A' + B' t_i \qquad (5.3.7)$$

If LINEST function is applied to the left side of formula (5.3.7) and the gradient and intercept are calculated by INDEX function, the result is as follows.

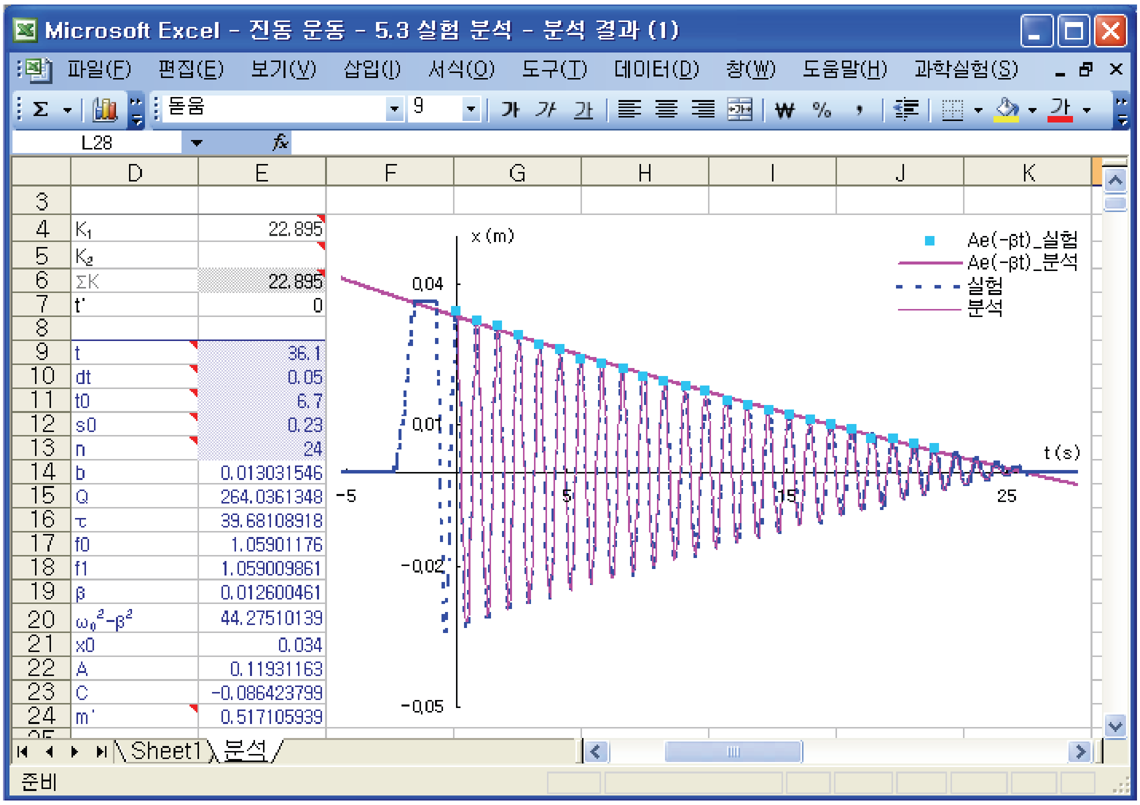
$$Y = \ln\left[\mathrm{GROWTH}(x_i - C)\right]$$

$$B' = \mathrm{INDEX}(\mathrm{LINEST}(Y),1)$$

$$A' = \mathrm{INDEX}(\mathrm{LINEST}(Y),2)$$

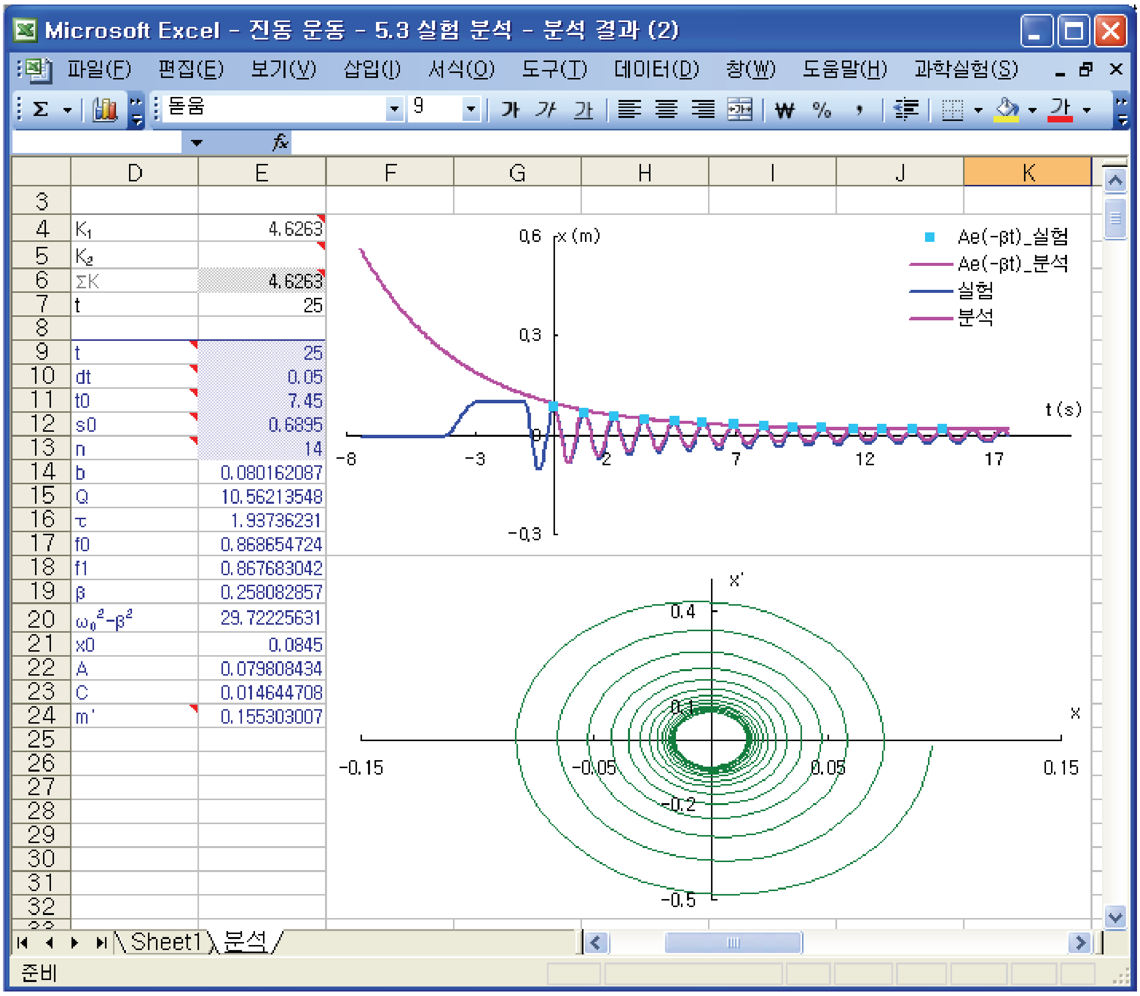
The result of fine tuning improves the accuracy of analysis just like in picture 5.3.7.

Picture 6.3.8 is the graph that shows the result of analyzing process. The experimental data contains the result of analysis and the analysis was done by taking the displacement x from the beginning till the equilibrium state of the experiment.



Picture 5.3.8 analysis result (1): The dotted line shows the curve of experimental data and the solid line shows the curve of the analyzed data. They are overlapping.

It the resistance of the system changes according to the initial conditions and physical circumstances, the motion of the system will be different by section[36], and the moduli A, C and β will be changed. If you observe the phase picture[37], you can see the width of damping change. Just like in picture 5.3.9, when a certain section of the data is observed the width of damping gets narrower as time passes by.



Picture 5.3.9 analysis result (2): When the damping resistance changes according to the time, you can see that the damping width of the phase picture gets narrower.

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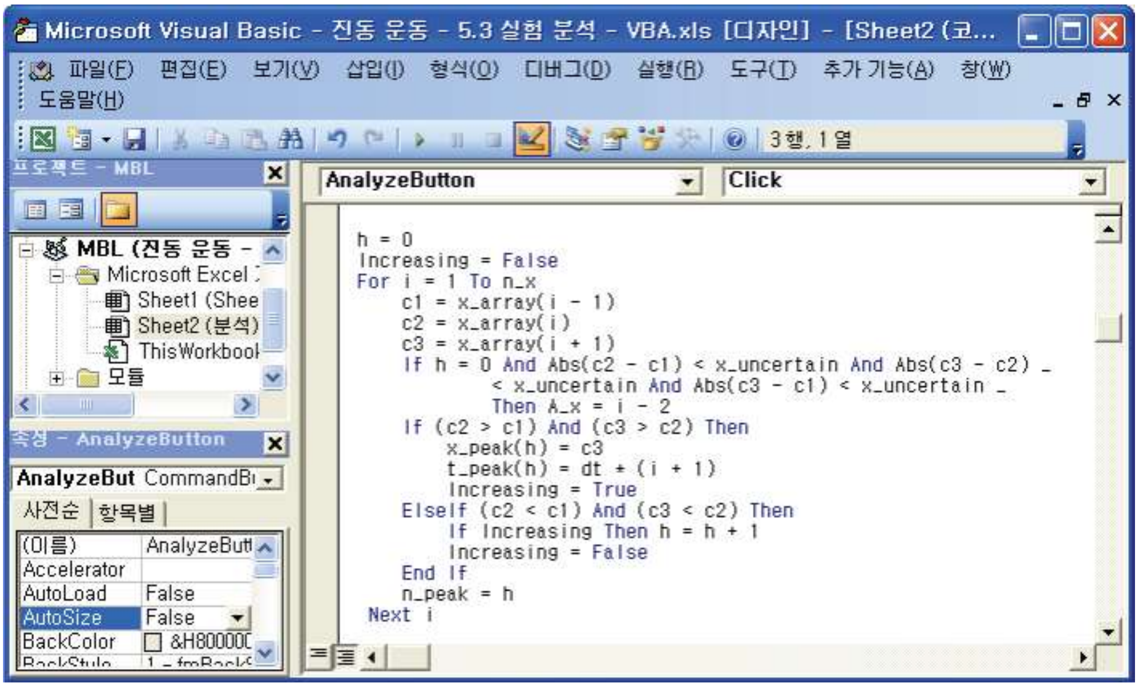
[36] If the damping resistance is not constant and changes as time goes by, it is difficult to analyze the experiment with only one damping constant. In this case, split the data by the time section analyze the experiment.

[37] Phase picture is the graph that express the real experimental data x_{Lab} with data x and $\dot{x}$ which are analyzed by the exponential curve fitting. x_{Lab} is recorded in column B of "oscillation.xls" file, and x in column B.

The real physical circumstances that have not been predicted or expressed until now can be analyzed accurately by fine tuning VBA codes. These series of explanation is about the mathematical methods of physical experimental analysis and if the analysis function of VBA is not used, the experiment analysis can be executed by using the analysis of physical experiment modeling which is explained in chapter 2.

VBA original code scene of picture 5.3.10 can be modified by choosing [Visual Basic Editor] of [Tool] menu and opening Private Sub Analyzebutton_Click() sub procedure whose order button is AnalyzeButton. The downloading site for this VBA original code[38] is introduced in the supplement of this book.



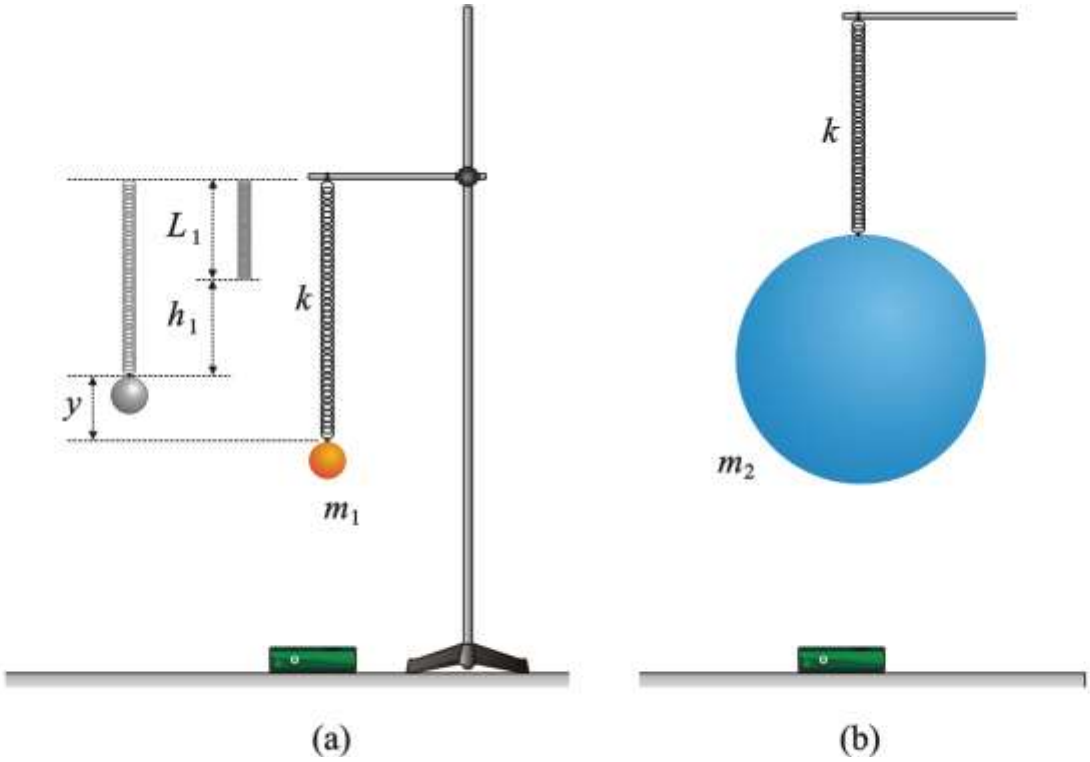
Picture 5.3.10 VBA window that has the experiment analysis codes[39]

[38] VBA original code is important for the students who study AP level physics or who are major in physics. However, this is not essentially relevant to the oscillation experiment, so it can be omitted.

[39] If you click AnalyzeButton, the data will be brought from the sheet and the result of the analysis will be recorded in “Analysis” sheet.

Exercise 5.3.1: The Factor That Affects the Damped Oscillation (1)

Analyze the motion when a spring that has the modulus of elasticity 4.62 hangs (1) a ball with a radius of 0.02m (2) a balloon with a radius of 0.18.



Picture 5.3.11 oscillation experiment of a spring pendulum with a radius of (a) 0.02m, (b) 0.18m

Set up the equation of motion and calculate the general solution for the spring pendulum which has periodic damping, just like the oscillation of picture 5.3.11. (a) is the case that the oscillation lasts for a long time because of the small damping, and (b) is the case that the damping is big because e of the resistance in the air.

When the resistance from friction is $b\dot{x}$, and the damping constant $\beta = b/2m$, the equation of motion for the spring pendulum is as follows.

$$m\ddot{x} + \beta\dot{x} + kx - mg = 0 \qquad (5.3.8)$$

And the relationship between the gravity and elasticity for the mass m is like below.

$$F_g = mg = -kh_1$$

And this can be rewritten as below when $\omega_0^2 = k/m$.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x - \omega_0^2 h_1 = 0 \qquad (5.3.9)$$

When the general solution which contains periodic damping is calculated with formula (5.3.9), the result is as follows.

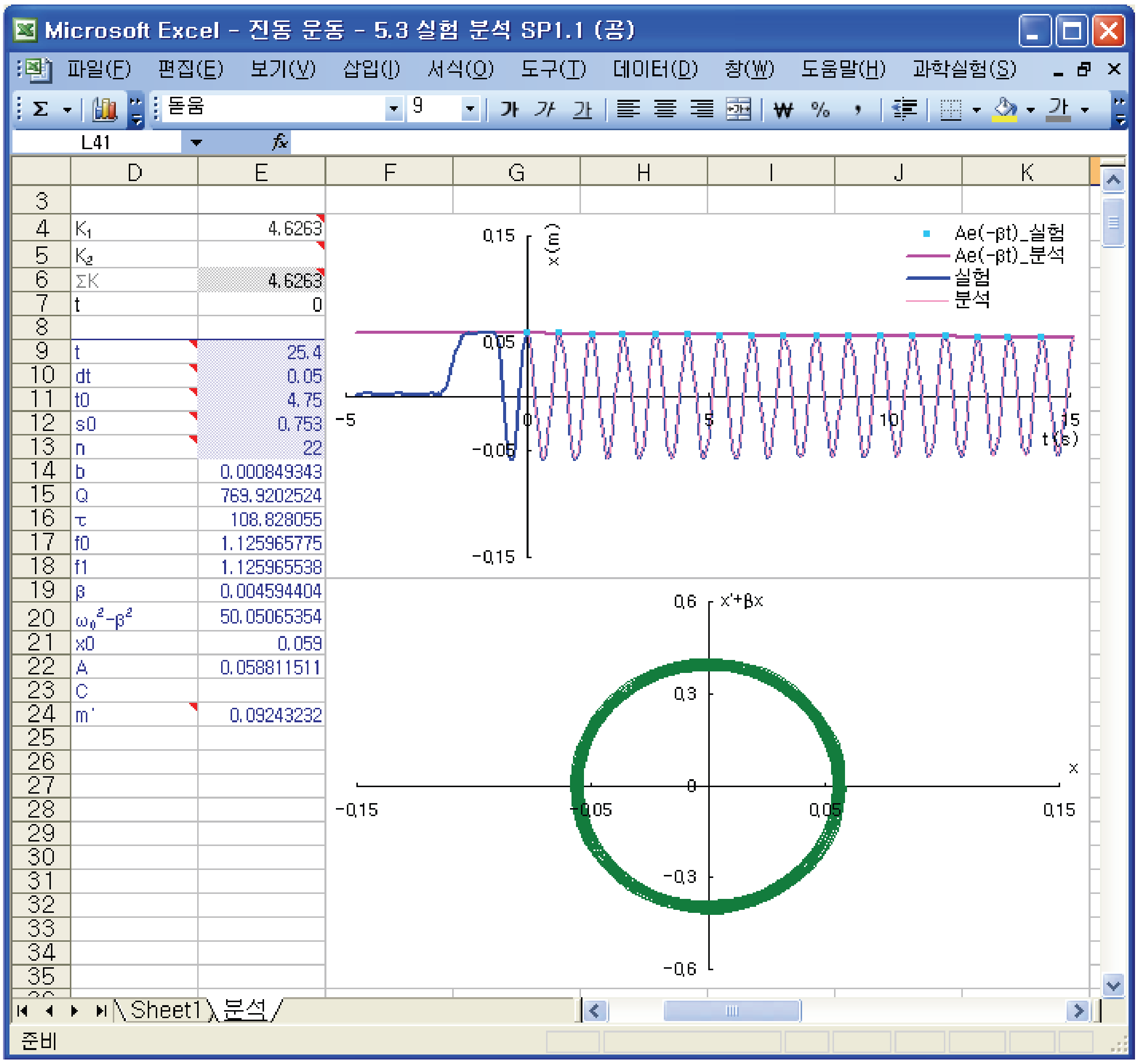
$$x = h_1 + Ae^{-\beta t}\cos\omega t \qquad (5.3.10)$$

	(a)	(b)
b	0.0008493	0.0255222
Q	769.9	38.7
τ	108.8	8.2
f_0	1.1259657	0.7461450
f_1	1.1259655	0.7460826
β	0.0045944	0.0606263
$\omega_0^2 - \beta^2$	50.05	21.98
x_0	0.059	0.105
A	0.0588115	0.0552523
C		0.0513199
m'	0.092	0.210

Table 5.3.2 results of the oscillation (a) and (b)

When analyzing the experiment, $L_1 + h_1$ of picture 5.3.11 should be 0 and calculate the displacement in the center of the oscillation. The results of experiment (a) and (b) in picture 5.3.11 are table 5.3.2. The result of analyzing (a)'s oscillation is picture 5.3.12, and (b)'s oscillation is picture 5.3.13. When you observe the damping related constant Q, β or b, you can see that the resistance is bigger in (b).

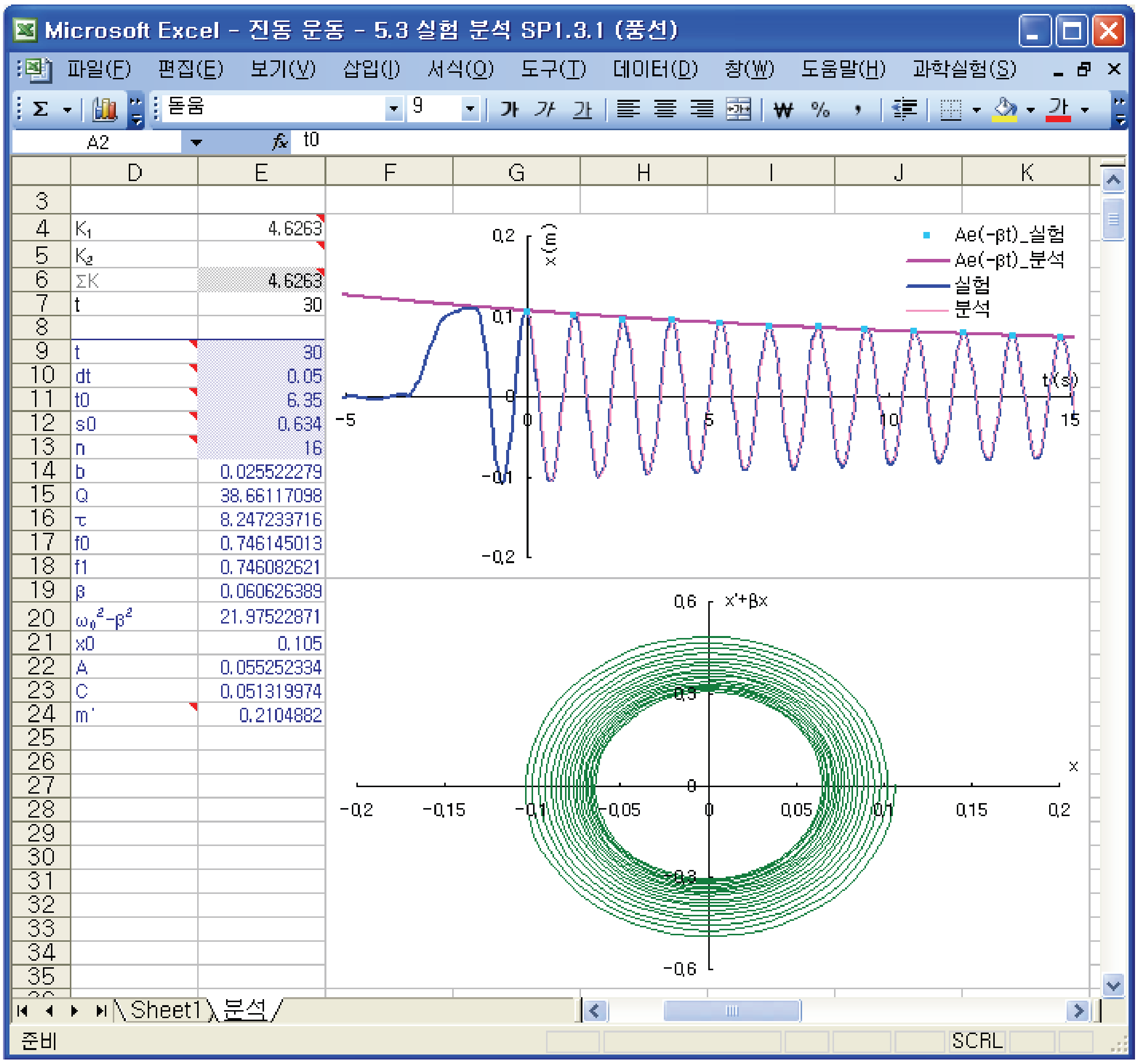
In case of (a), the damping resistance is small, so in the $x-t$ graph of picture 5.3.12, you cannot discern whether the amplitude gets smaller or not by the damping, but in case of (b), you can easily see that the amplitude gets smaller in the graph. Also, in case of (b), when you observe the $\dot{x}+\beta t$ graph, you can see that the width of the amplitude's decrease gets smaller as time goes by. The natural frequency of (a) f_0 is 1.1259657 but the damping frequency f_1 is 1.1259655 and $\Delta f = f_0 - f_1 = 0.0000002$, so when the damping resistance is small, $f_0 \approx f_1$. In the section where the damping is big, (b) has shorter time constant r than (a). (a) has so small resistance that it shows the graph similar to the free oscillation.



Picture 5.3.12 graph of the experiment with the ball with a radius of 0.02m

The experiment mass m' is the result of adding parts of the oscillating spring's mass to the galls mass, and this can be calculated accurately when you know the spring constant K.

In the displacement graph of picture 5.3.13, the curve of damping amplitude is definitely curved. In the phase graph of displacement and velocity, the gap between lines gets narrower as it enters from the outside to the inside. This result shows the circumstance that has a big damping because of the air resistance[40].

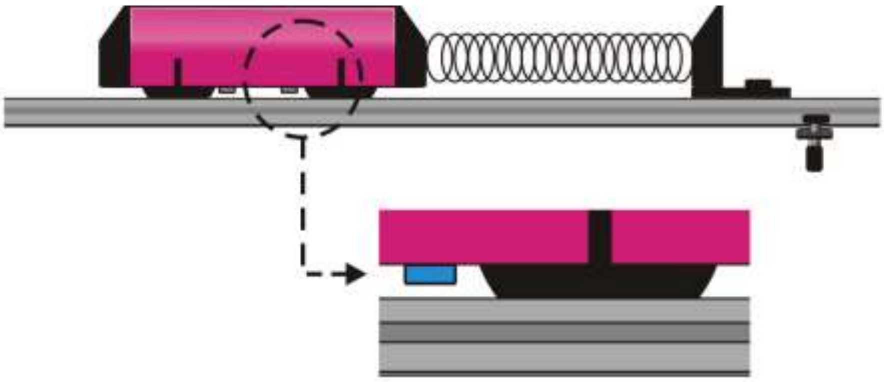


Picture 5.3.13 result of the experiment with a balloon with a radius of 0.18m: $x-t, x-\dot{x}$ graphs

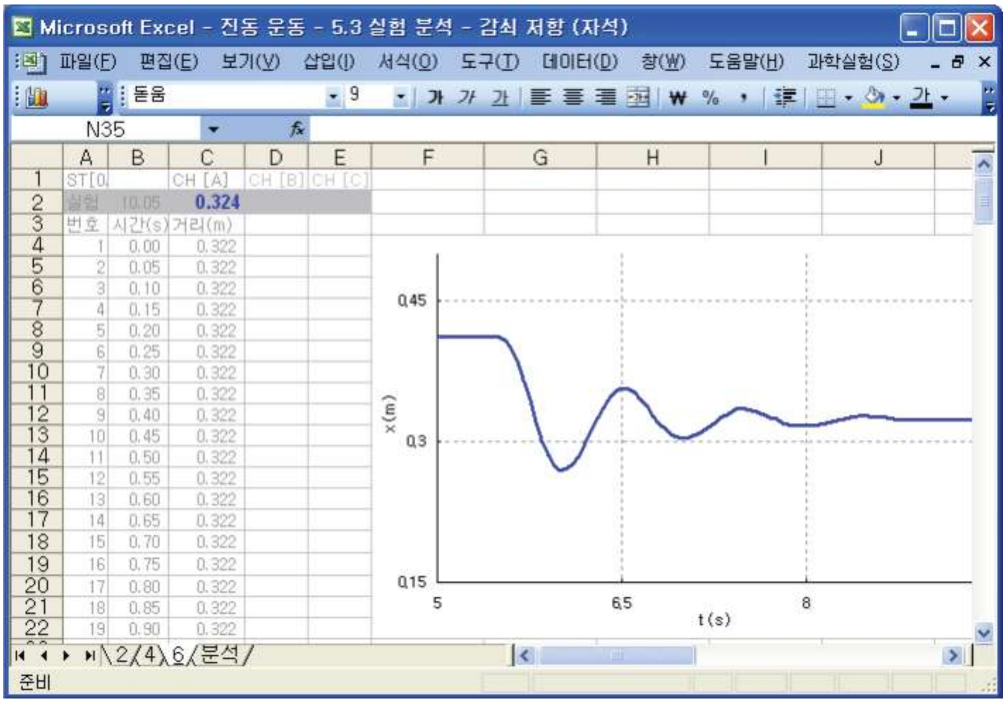
40 The system is affected by the force directly proportional to the velocity caused by the air resistance. Because the velocity of a balloon is snow, so add $f=-bv$ and predict and analyze the modeling.

Exercise 5.3.2: The Factor That Affects the Damped Oscillation (2)

Analyze the amplitude graph of the damped motion when a magnet is attached to the bottom of a cart.



Picture 5.3.14 motion of a cart which gets the damping resistance by a magnet[41]



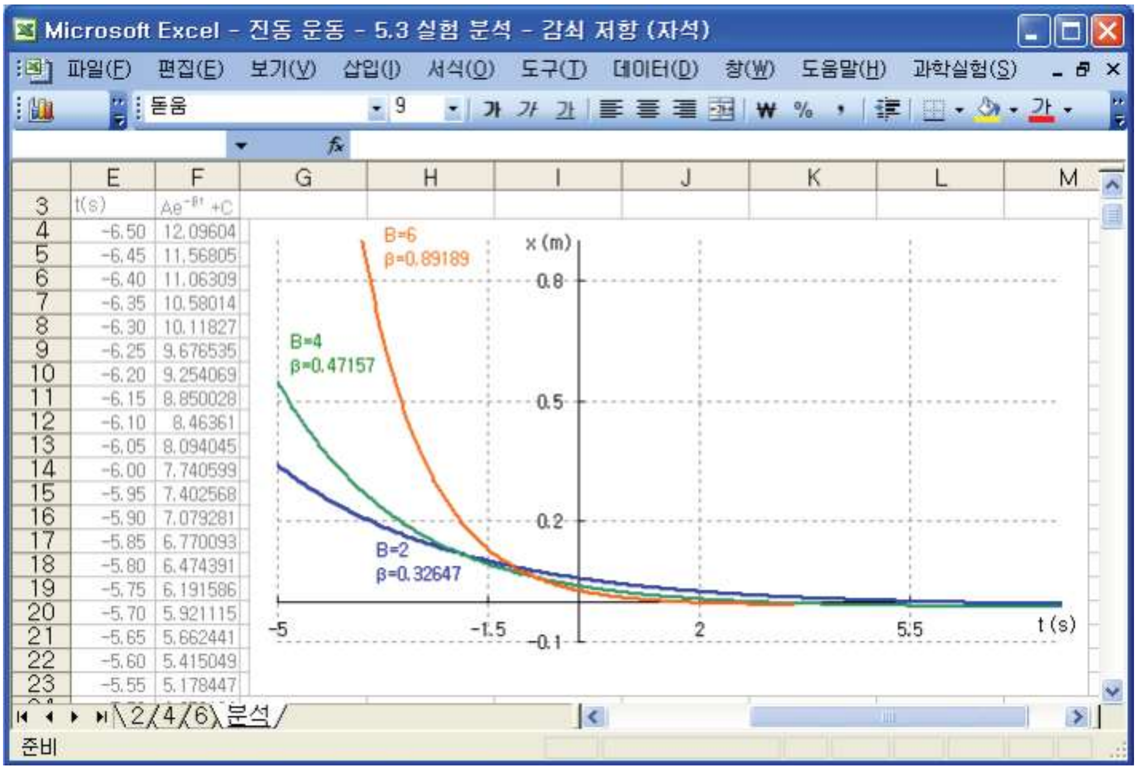
Picture 5.3.15 $x-t$ x graph of the damped motion experiment when six magnets are attached

[41] When the cart moves, the changes in the magnetic field caused by the magnetic generate the eddy current to the surface of the track. This current generates the magnetic filed in the direction that disturbs the magnet's motion, so it takes the role of damping resistance for the cart's motion.

As the cart moves, it gets the damping resistance by the magnet, so this resistance makes the damping constant bigger. When the magnets are attached 2, 4, and 6 for each case, the amplitude graphs are like picture 5.3.16. In table 5.3.3, the moduli of the amplitude curve are calculated and expressed from each graph. For example, when the magnets are 6, the amplitude graph is $x = 0.03675e^{-0.89189t} - 0.00975$. With the graph and the table, you can see that the damping resistance for the cart gets bigger when the number of the magnet increases.

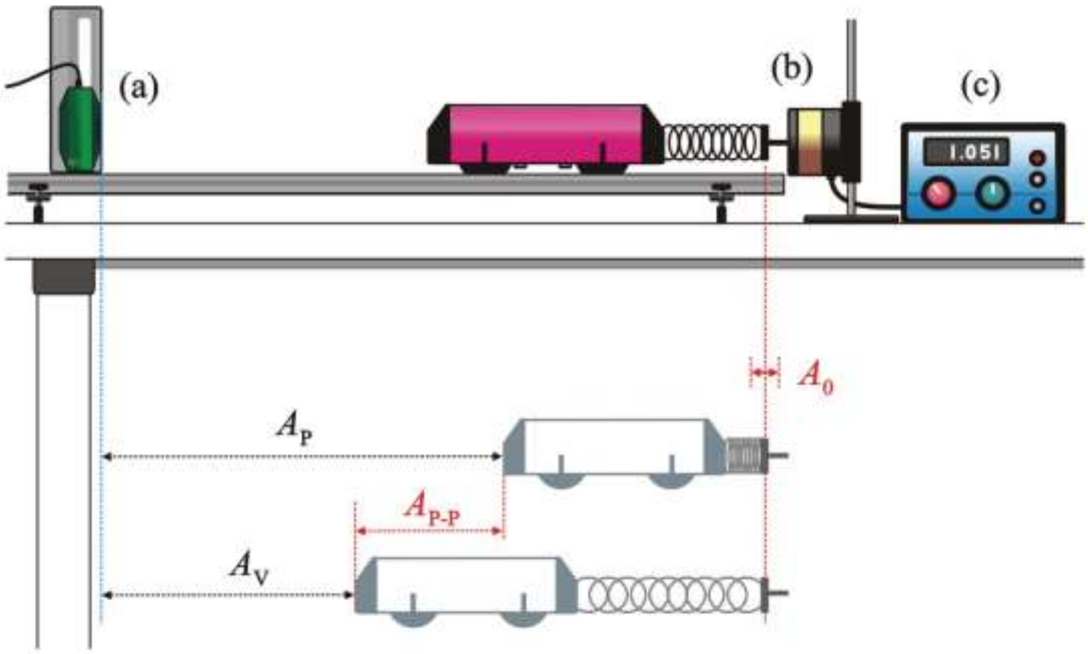
	B=2	B=4	B=6
β	0.32647	0.47157	0.89189
A	0.06854	0.05322	0.03675
C	-0.01112	-0.01472	-0.00975

Table 5.3.3 damping constants calculated from the oscillation of the cart that has magnet



Picture 5.3.16 amplitude graph of a cart that has damped motion because of magnets: the graph using table 5.3.3

5.3.3 Analyzing Process – Forced Oscillation



Picture 5.3.17 forced oscillation experiment using dynamic oscillator: (a) motion sensor (b) dynamics oscillator (c) function generating device

Let's conduct a forced oscillation experiment using dynamic oscillator just like picture 5.3.17. When the range of $n = \omega/\omega_0 = f/f_0$ on the natural frequency ω_0 is bigger than 1 and smaller than 1, if you do the experiment which calculates the amplitude A in the normal state per regular interval, you can analyze the experimental data of A/A_0 on n and draw it as a graph. ω is the frequency forced by the function generating device and ω_0 is the natural frequency. Include the modulus of elasticity K, the cart's mass M and the additional mass m, and calculate $\omega_0 = \sqrt{K/(M+m)}$ [42]. A_0 is the P-P(peak to peak) displacement[43] of the dynamic oscillator from the natural frequency. When you cause the forced oscillation, the amplitude A should be measured by a motion sensor. The amplitude A can be expressed by measuring $A(p-p)$ expressed in P-P displacement.

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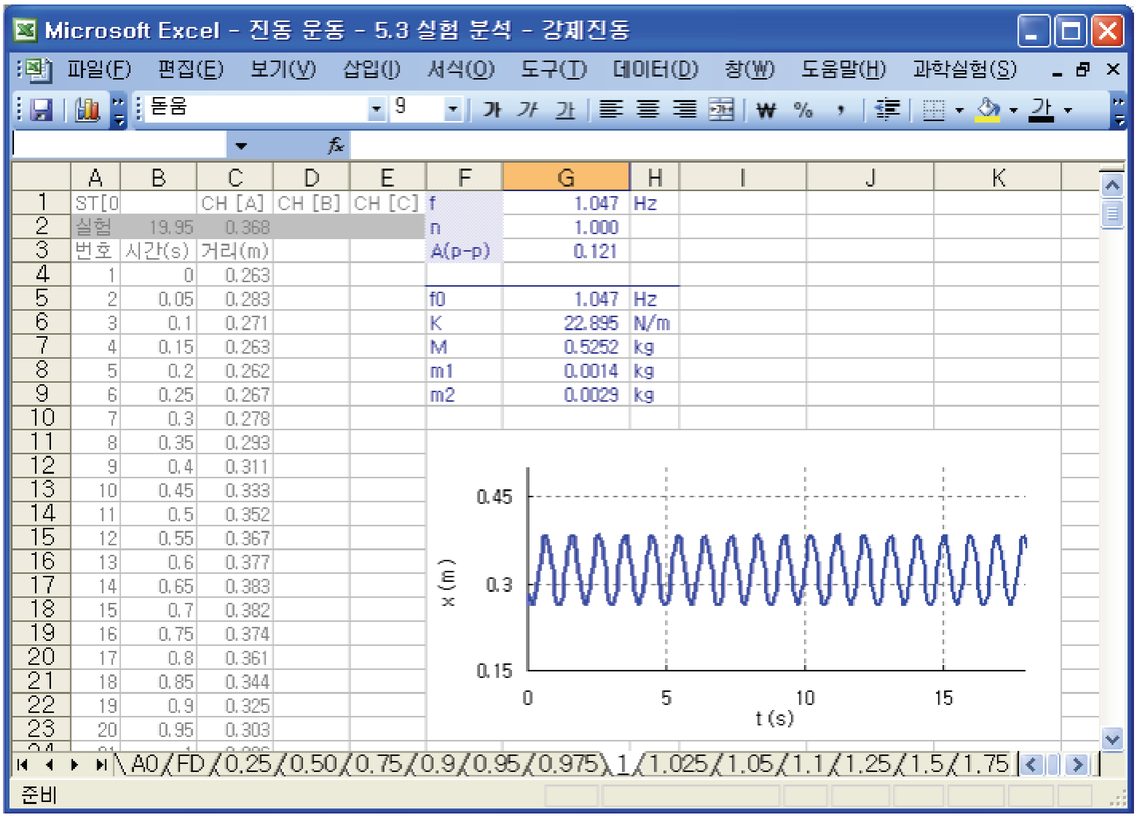
[42] The additional mass contains the adhesive tape, magnets and so on.
[43] amplitude that indicates the length between the peak and the valley in a oscillation

	Cell	Formula or Value
f	G1	=G5*G2
n	G2	=G5*G2
$A(\mathrm{p-p})$	G3	=+ MAX(C10:C400)-MIN(C10:C400)
f_0	G5	=+ SQRT(G6/(G7+ G8+ G9))/(2*3.141592)
K	G6	Modulus of elasticity
M	G7	Mass of cart
m_1	G8	Mass1
m_2	G9	Mass2

Table 5.3.4 values and formula to be applied to the cells of worksheet in the forced oscillation experiment

The experiment analysis can be done by applying these formulae. The detailed values and formulae are in table 5.3.4. When calculating P-P amplitude, the formula range[44] to calculate the gab between the maximum and minimum values should exclude the beginning and end data of column C[45]. This is done for eliminating the uncertainty of data in the beginning and the end of the experiment. Picture 5.3.18 is the result of applying the values of table 5.3.4 to the cells of worksheet and calculating P-P amplitude of the forced oscillation in the sheets of n0.25, 0.5, 0.75, 0.9, 0.95, 0.975, 1, 1.025, 1.05, 1.1, 1.25, 1.5 and 1.75. The results of each sheet should be analyzed in “Analysis” sheet.

[44] When the measuring interval is 0.05 second, data collecting time is within 20 second, total number of data will be 400. When the data are 400, they will be recorded from C4 to C 403.
[45] In table 5.3.4, the formula range of A(p-p) was set up “C10:C400”.



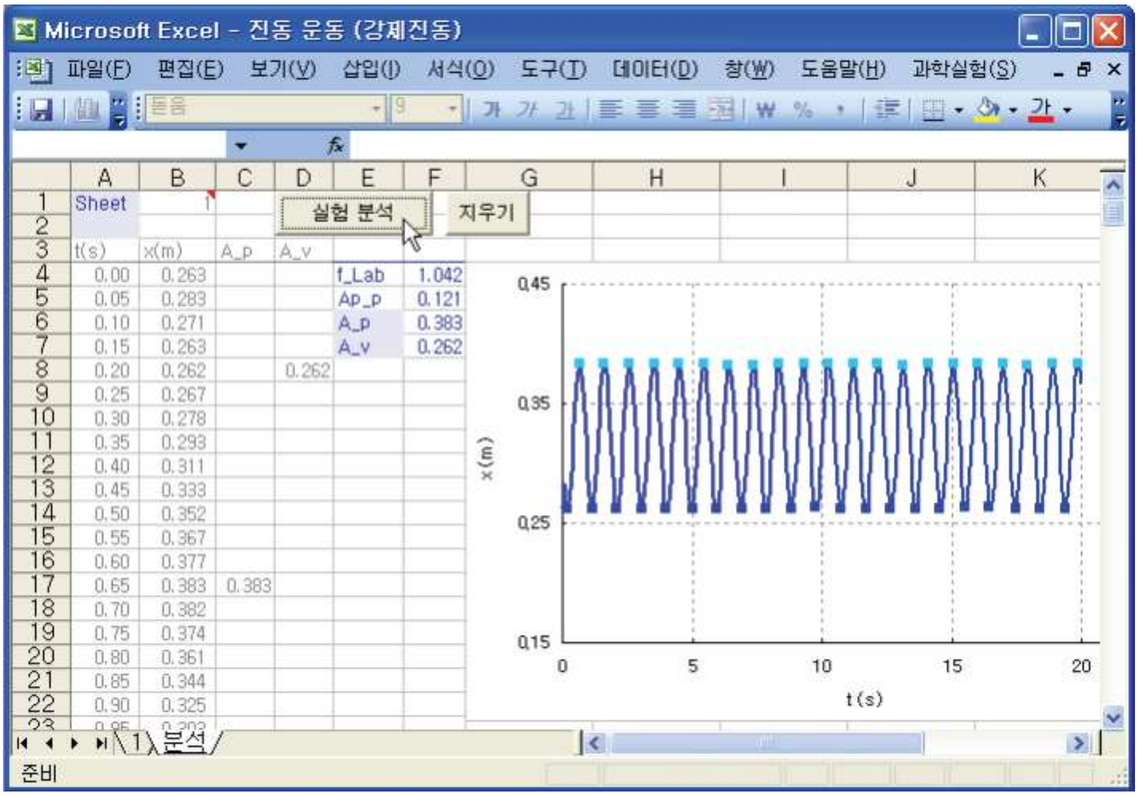
Picture 5.3.18 data collecting and experiment analysis for the forced oscillation experiment: Calculate P-P amplitude A(p-p)[46] in the worksheet that collected the experiment data.

The amplitude A(p-p) calculated in the cell G3 is the gap between the maximum and minimum values of the data collected in column C, so let's check out how this value is different with the average of P-P amplitude within the error range.

Picture 5.3.19 is the graph of normal state when the forced oscillation is caused with frequency f+1.047Hz. According to the formula calculation of table 5.3.4, A(p-p)=0.121m can be acquired. Calculate the gap between the average of peaks and average of valleys by selecting peaks and valleys in the experimental data of picture 5.3.18. Compare this value with the value calculated with the maximum and minimum values of the amplitude. Picture 5.3.19 is the worksheet of "Oscillation(Forced Oscillation)" file to calculate P-P amplitude form the average of peaks and valleys. If

[46] The calculating formula for P-P amplitude has been written in cell G3, so the result can be acquired at the same time.

you click [Experiment Analysis], the experimental data is brought from the sheet that has written the name in cell B2, peaks and valleys are analyzed, and f_{Lab}[47], which is the experimental frequency of the forced oscillation and A(p-p) are calculated.

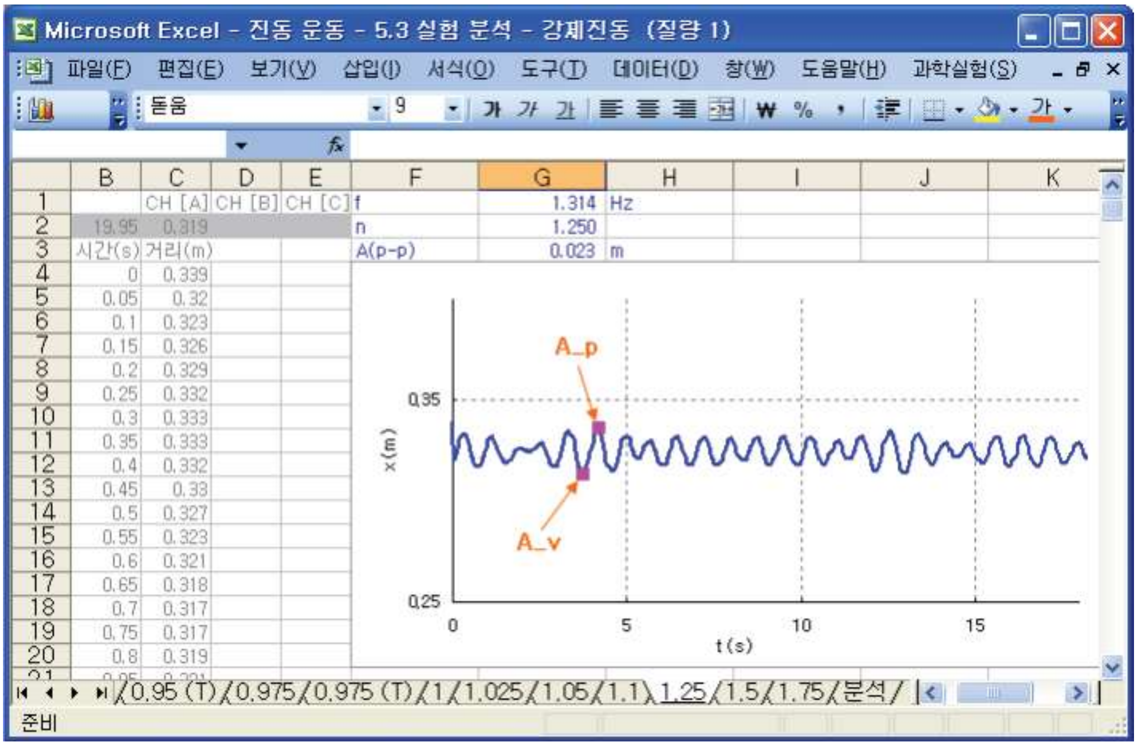


Picture 5.3.19 calculating P-P amplitude of forced oscillation: Calculate P-P amplitude from the gap between the average of peaks and valleys

When the error range of the motion sensor is ±0.002m, within the error range, the value calculated in picture 5.3.18 is the same as the value A(p-p)=0.12067 calculated in picture 5.3.19. So in the forced oscillation experiment, when data is collected after the cart enters in the normal state, if the error of P-P amplitude is small just like in picture 5.3.18, it is possible to calculate P-P amplitude with the maximum and minimum values of the experimental data. However, if n is much smaller than 1 or much bigger than 1, that is, if the forced oscillation amplitude is extremely small, the motion uncertainty of the cart increases as in picture 5.3.10, so P-P amplitude should be calculated with the gap of average values just like in picture 5.3.19.

---

[47] The experimental frequency f_{Lab} is different from the frequency $f = nf_0$ (which is the frequency of dynamic oscillator) within the error range.

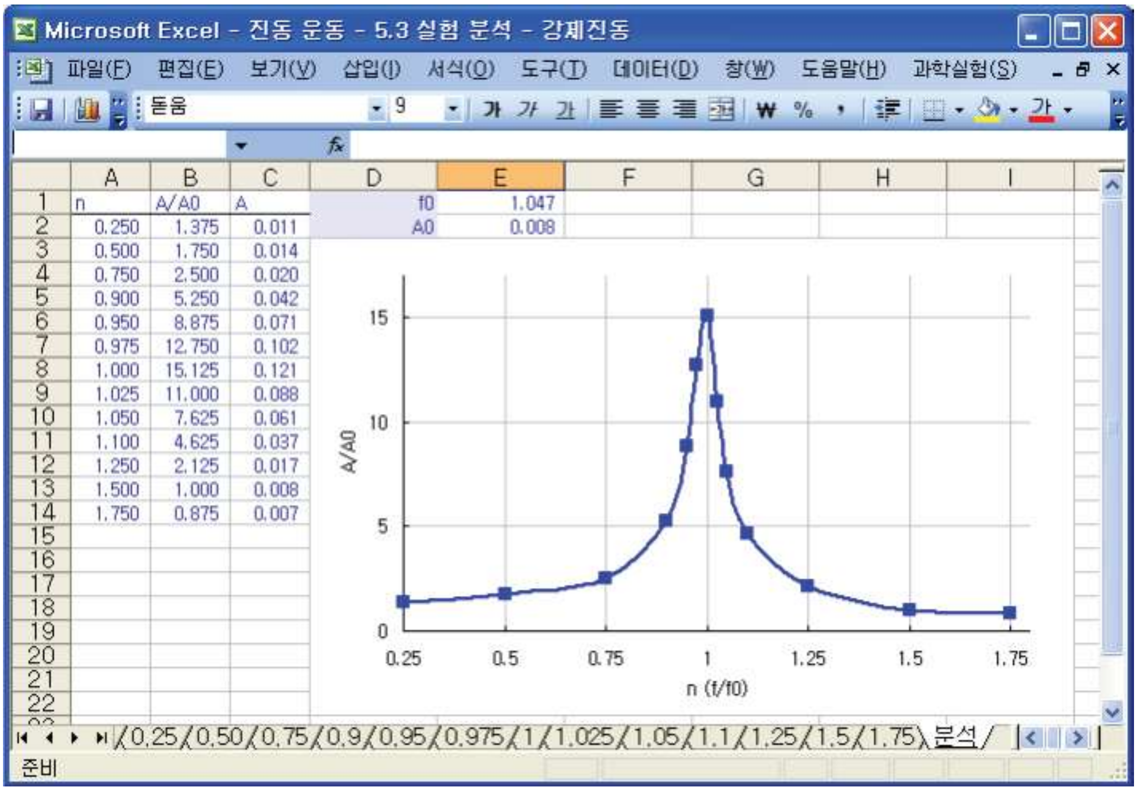


Picture 5.3.20 experiment of a cart that has a big motion uncertainty[48]: In this case, analyze A(p-p) with “Oscillation(Forced Oscillation).xls” file [49] just like picture 5.3.19.

Based on this result, let’s draw a graph with the ratio of P-P amplitude A/A_0 and the ratio of frequency $n = \omega/\omega_0 = f/f_0$. They are expressed by the Forced Harmonic Oscillator in a normal state. Picture 5.3.21 is the result of a forced oscillation caused by a cart that has damped resistance because of a neodymium magnet. In this experiment, when the number of the magnet increases, the Q-constant gets smaller and the peak of the graph gets lower in the graph. Also, based on this result, formulae for physical models can be set up and the prediction and analysis can be executed.

[48] The A(p-p) formula calculation of cell G3 is the same as the maximum(A_p) – the minimum(A-v) of the amplitude.

[49] The original VBA code of Oscillation(Forced Oscillation).xls file is in the supplement.

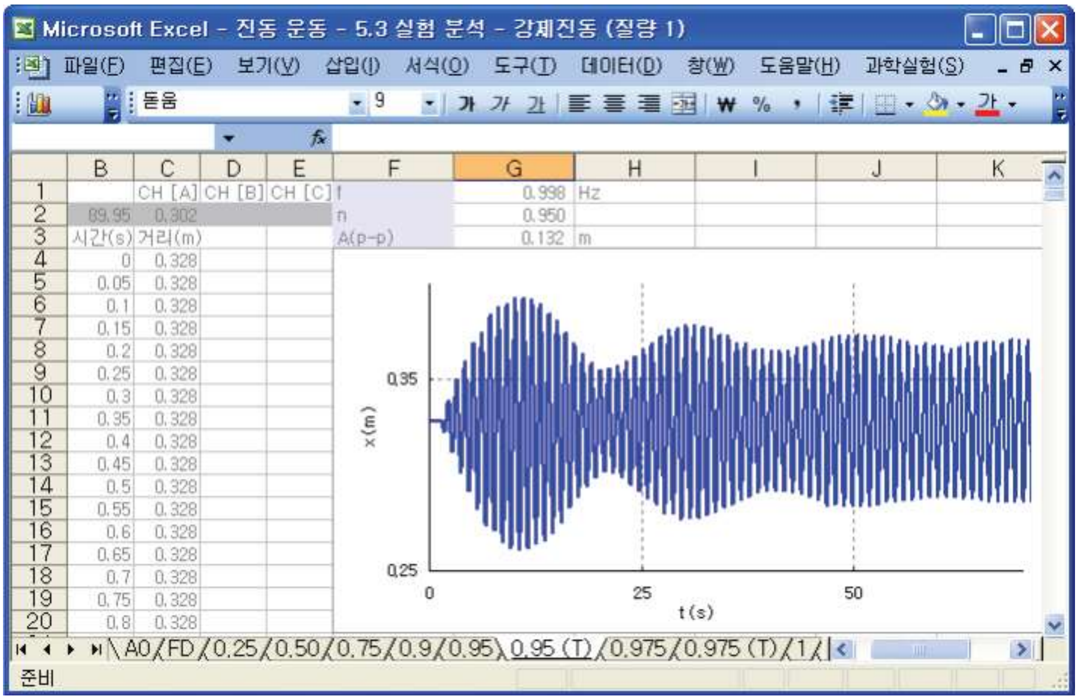


Picture 5.3.21 graph of A/A_0 and f/f_0 in a forced oscillation experiment of a cart that has damped resistance caused by magnets

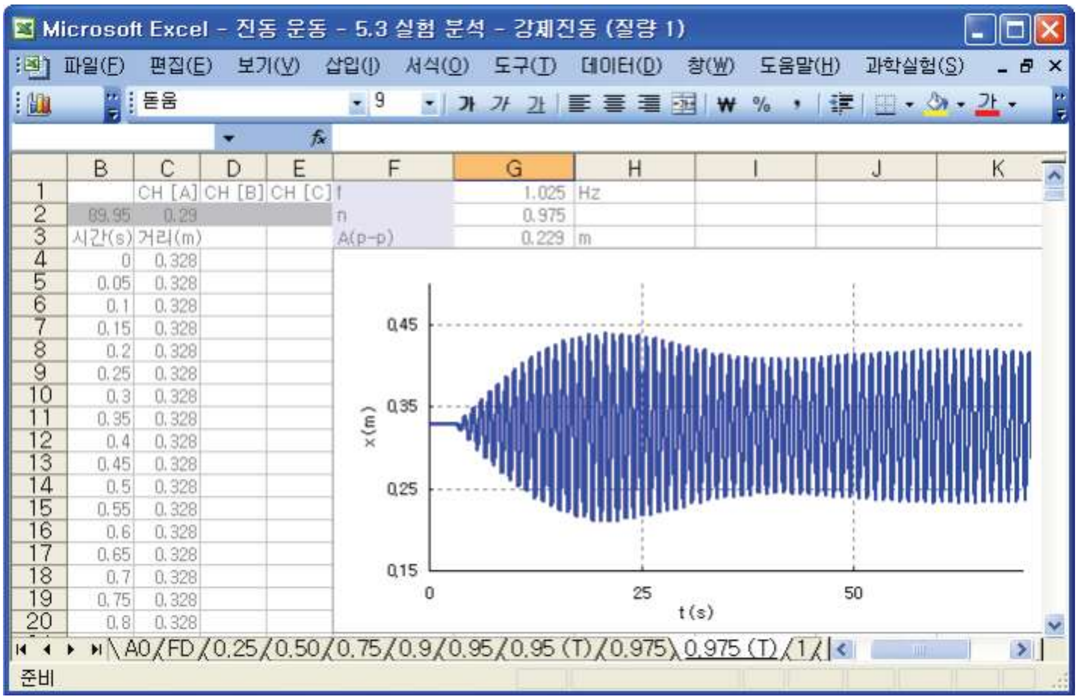
---

Exercise 5.3.3: The Transient State Expressed by Forced Harmonic Oscillator

Let's calculate the amplitude graph for the early section in which FHO is in the transient state. In this state, the changing shapes of the amplitude are various. As below, let's find out the amplitude graph of early state in various experimental circumstances.



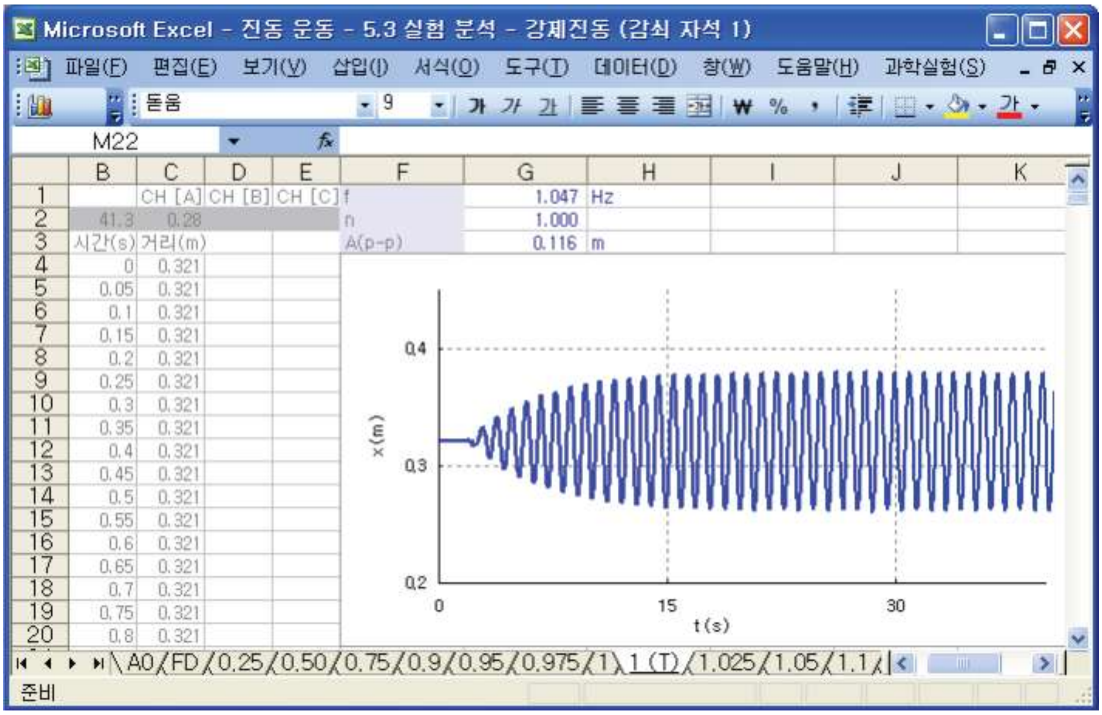
Picture 5.3.22 experiment graph of a cart's oscillation when Q=27, n=0.95[50]



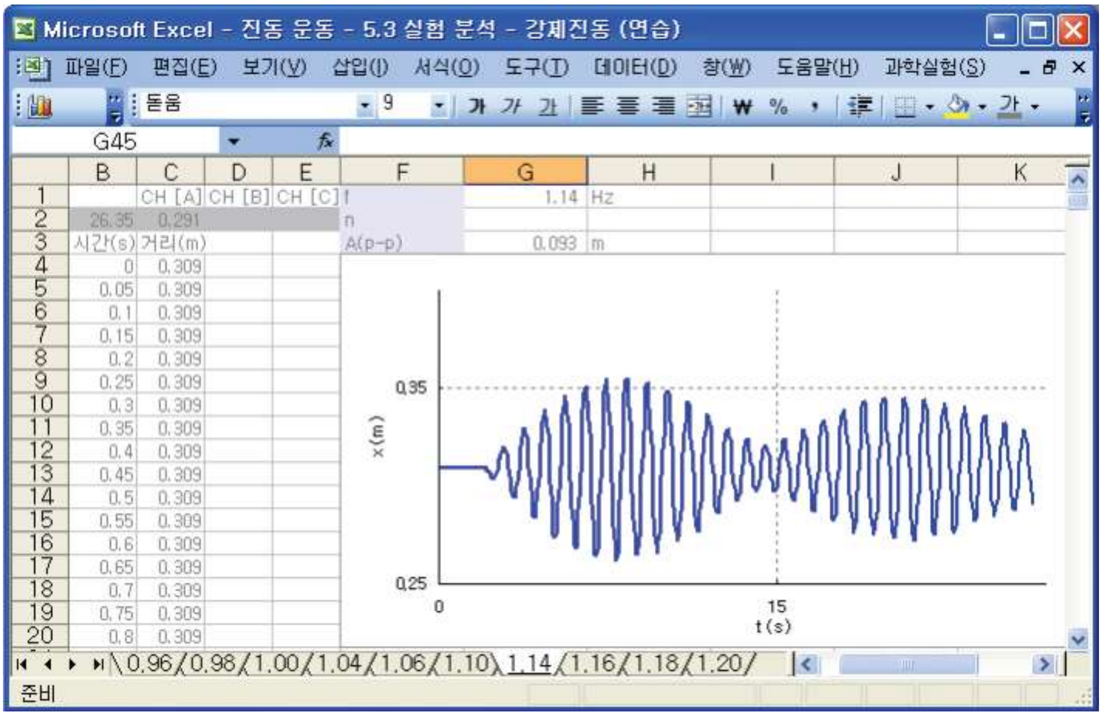
Picture 5.3.23 experiment graph of a cart's oscillation when Q=27, n=0.975[51]

50 mass m=0.525kg, modulus of elasticity K=22.895N/m, f=0.998Hz, f0=1.051Hz, A(p-p)=0.132m

51 mass m=0.525kg, modulus of elasticity K=22.895N/m, f=1.025Hz, f0=1.051Hz, A(p-p)=0.229m



Picture 5.3.24 experiment graph of a cart's oscillation when Q=15, n=1[52]



Picture 5.3.25experiment graph of a cart's oscillation when f=1.14Hz[53]

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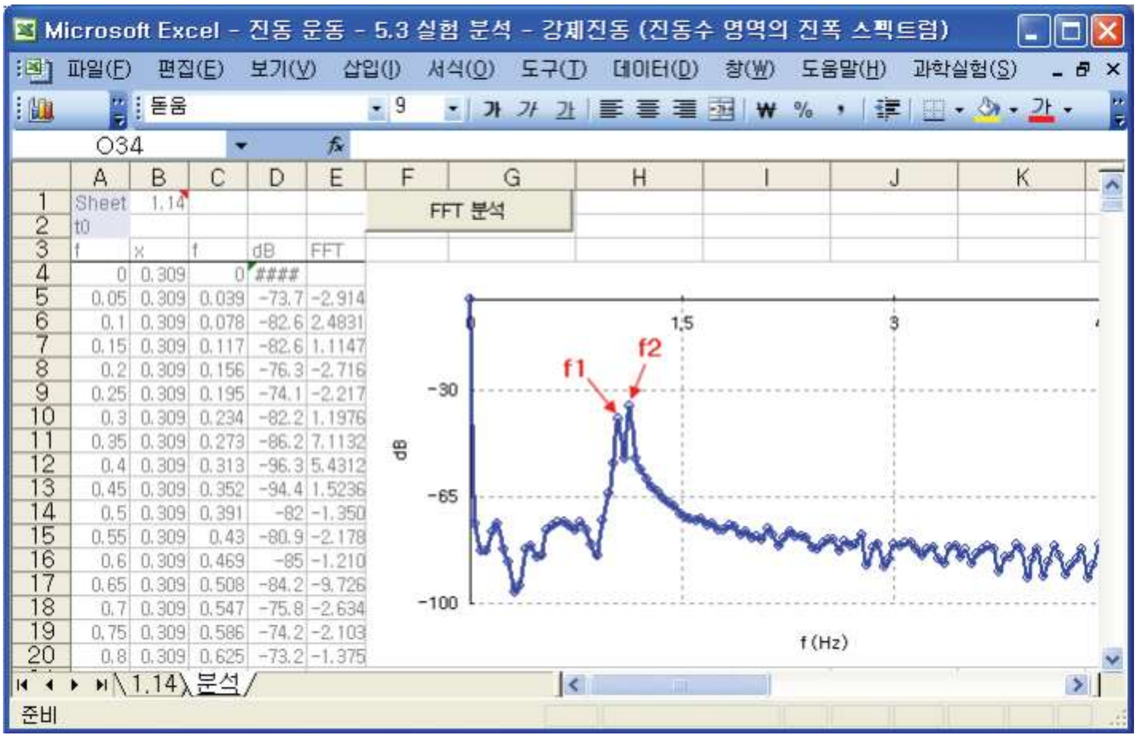
[52] mass m=0.525kg, modulus of elasticity K=22.895N/m, f=1.047Hz, f0=1.047Hz, A(p-p)=0.116m

[53] This is done with an arbitrary frequency when f0 is unknown.

Picture 5.3.25 shows that the two neighboring frequencies overlap each other in the transient state. Picture 5.3.26 is the frequency spectrum graph drawn by analyzing the oscillation of a cart. In this graph, the overlapped frequencies in the transient state just like the two peaks can be calculated by analyzing FFT amplitude spectrum[54].



Picture 5.3.26 amplitude spectrum analysis within the frequency range in the experiment of picture 5.3.25: calculating two overlapping frequencies in forced frequency f=1.14Hz

Exercise 5.3.4: Graph of Frequency and Amplitude for the Forced Oscillation Experiment

IN the forced oscillation, when the outer force and mass is constant, let's draw the graph of amplitude A/A_0 and frequency f/f_0 that change according to the damping constant β. Table 5.3.5 is the result of the experiment in which a cart with a mass of 0.525kg executes forced oscillation by a constant force caused by the dynamic oscillator.

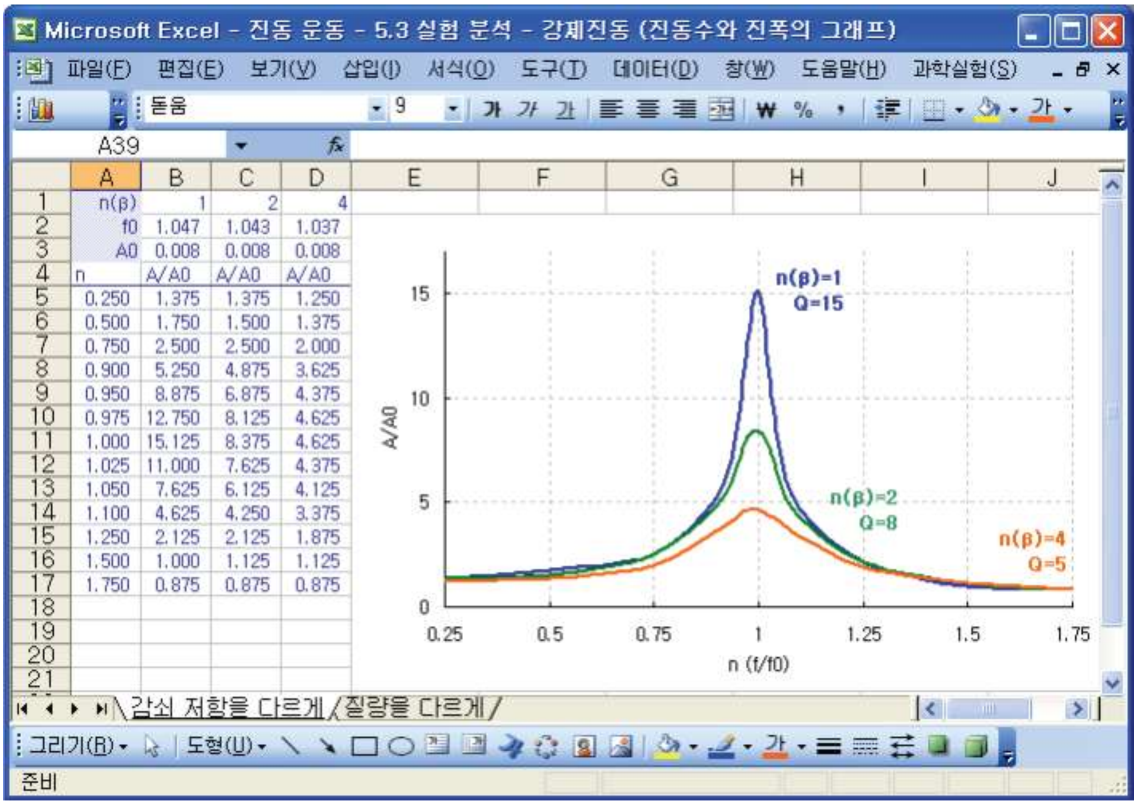
[54] Excel VBA original code to analyze FFT amplitude spectrum within the frequency range is introduced in detail in chapter 2 and the supplement.

Damping	n(β)=1	2	4
f_0	1.047	1.043	1.037
A_0	0.008	0.008	0.008
0.25	1.375	1.375	1.250
0.5	1.750	1.500	1.375
0.75	2.500	2.500	2.000
0.9	5.250	4.875	3.625
0.95	8.875	6.875	4.375
0.975	12.750	8.125	4.625
n= 1.0	15.125	8.375	4.625
1.025	11.000	7.625	4.375
1.05	7.625	6.125	4.125
1.1	4.625	4.250	3.375
1.25	2.125	2.125	1.875
1.5	1.000	1.125	1.125
1.75	0.875	0.875	0.875

Table 5.3.5 results of forced oscillation experiment according to the damping constant

Picture 5.3.27 is the graph of amplitude A/A_0 and frequency f/f_0 drawn by the result of table 5.3.5. The amplitude A_0 caused by the dynamic oscillator's force[55] is 0.08m. The neodymium magnet is used as a damping resistance and this is the result of the experiment when the number of the magnet is 1, 2, and 4. As the magnets increase, the damping resistance increases, the height of the amplitude's peak gets lower and the sharp shape gets smoother. We can guess by the experiment analysis that the reason why the height and shape of the peak get changed is because of the size of damping resistance.

[55] P-P amplitude should be measured with no devices attached to the dynamic oscillator,



Picture 5.3.27 graph of amplitude A/A_0 and frequency f/f_0 in the forced oscillation experiment: The graph is drawn by the result data of the experiment when n=from 0.25 to 1.75.

Compare this graph with the graph drawn in the theory. By writing mathematical formulae for physical models and analyzing the result data, compare and predict theoretically[56].

Experiment analysis process in chapter 5 is the analysis process using “Oscillation.xls” file and this is for the simple analysis with [Experiment Analysis] button to which VBA is applied. However, without this file, you can execute the experiment in Sheet1 and analyze the experiment by the data analysis based on physical modeling in chapter 2. You can do custom educations according to the teaching circumstances.

[56] Refer to data analysis based on physical modeling in chapter 2.

5.4.

Experiment: Free Oscillation

5.4.1. Experiment Outline

Using the oscillation experiment of a spring pendulum which is close to the free oscillation, let's calculate the location, velocity and energy of the pendulum according to time and analyze the frequency and damping constant so that we can find out the causes which affect the damping of the pendulum. By this, the physical concepts concerning free oscillations and damped oscillations can be understood.

Goal

Understanding characteristics of the spring pendulum's oscillation

Required Equipments

Electronic scale	1
Motion sensor	1
Measuring tape	1
Springs (of different lengths)	3
Pendulums (50g)	5
Ball (of big volume and small volume)	1

Picture 5.4.1 oscillation experiment of a spring pendulum: motions of a big ball and a small ball that have different masses and volumes

5.4.2 Experiment A: Measuring a Spring's Modulus of Elasticity

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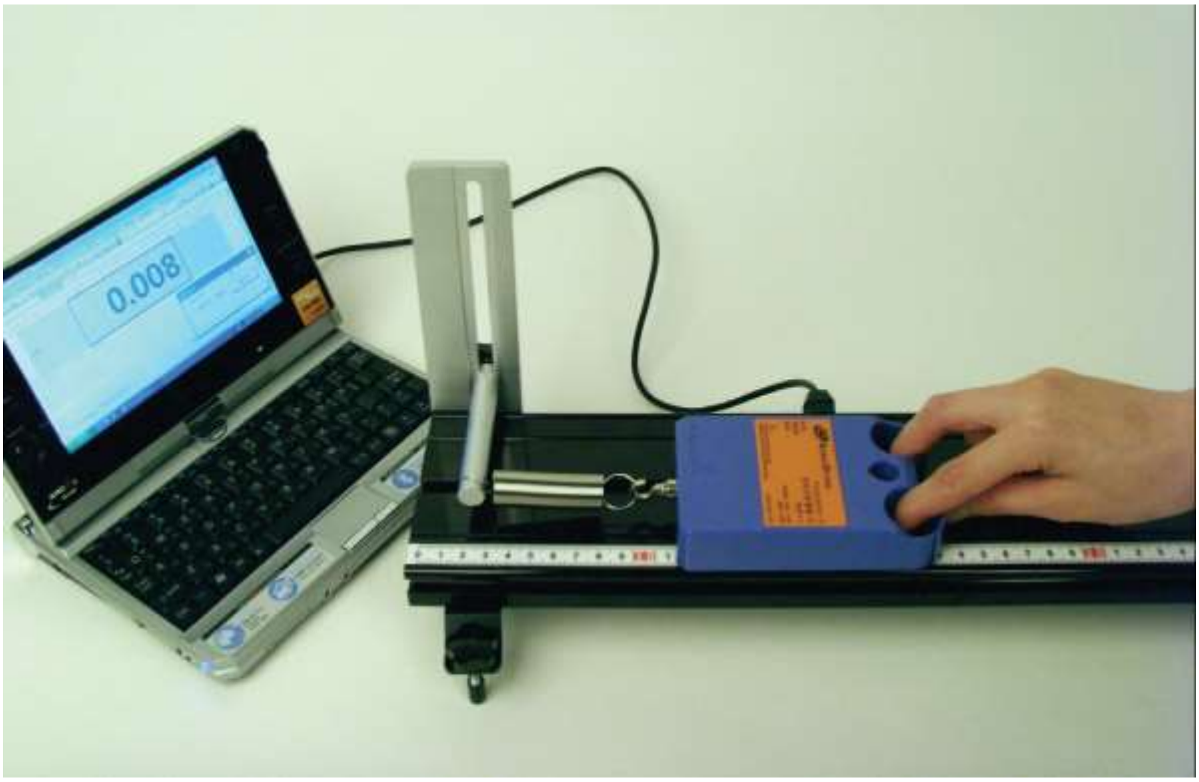
Experiment Prediction: the modulus of elasticity according to the spring's length

1. Let's measure the modulus of elasticity by cutting the 20cm spring in the ratio of five to one.

   a. What will be the shorter spring's modulus of elasticity?

   b. How will the modulus of elasticity change when the two same length springs are put together?

Experiment Process:



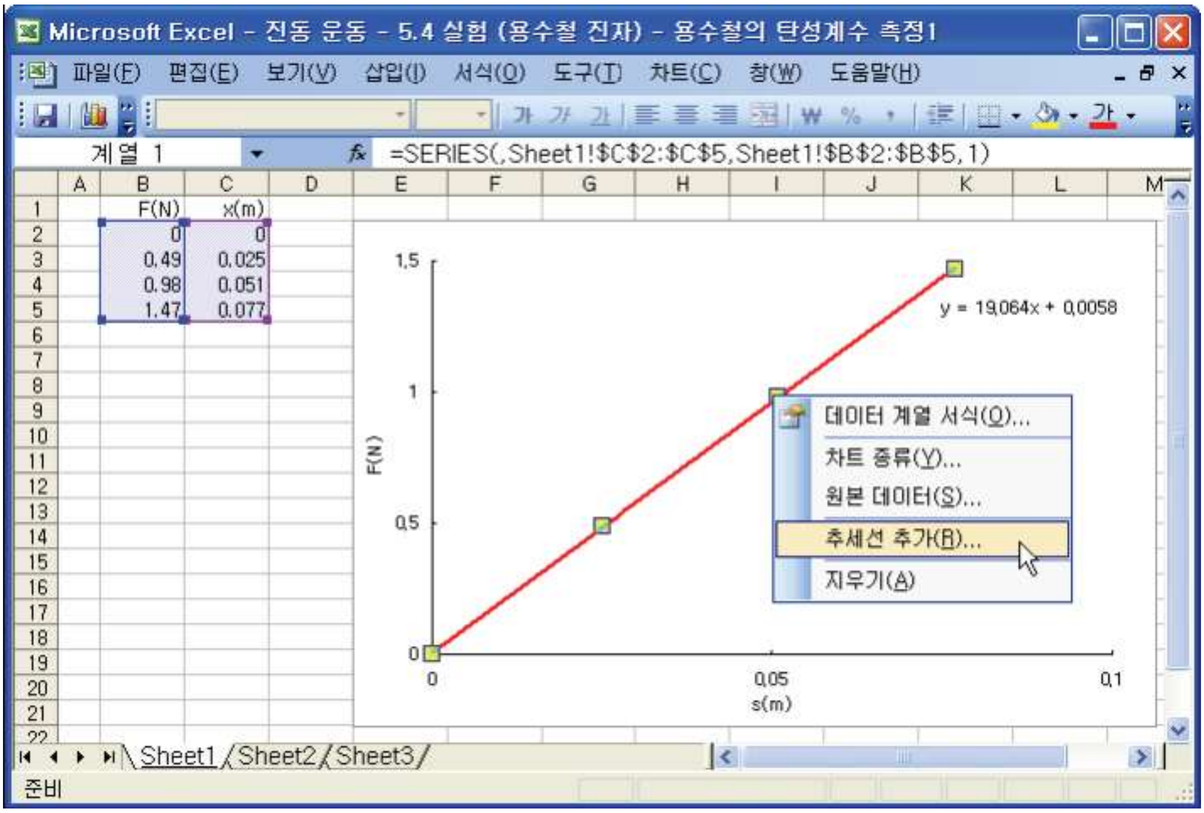
Picture 5.4.2 measuring the spring's modulus of elasticity: this is the way of measuring by load cell. Fix one end of the spring and observe the left end of the load cell on the scale of the cm ruler and expand the spring to a certain length. Measure the size of the force and calculate the modulus of elasticity.

1. Prepare a computer as in picture 5.4.2 and connect the load cell to channel A.
2. Open [Science Cube]-[Experiment] window in the menu of Excel worksheet. Cancel [Data Recording in Cell] and click [Start Experiment], then the measured data will be recorded in cell C2.
3. Read the value of cell C2 when the spring is expanded by the load cell to a certain length.
4. Draw a graph about the force and the expanded length[57] as in picture 5.4.3 and calculate the modulus of elasticity in the trend line formula.

[57] Subtract the scale's first value from all the scale values so that the origin of the graph passes (0, 0).

5. Put two springs together as a series and repeat 1, 2 to measure the modulus of elasticity.

6. Calculate the modulus of elasticity by using various pendulums whose masses are known, instead of the load cell[58].



Picture 5.4.3 Excel scene calculating a spring's modulus of elasticity: Add a trend line to the graph about the expanded length of the springs when pendulums are hung to them. Calculate the gradient k with the trend line formula $y = kx + b$, and then the spring's modulus of elasticity can be acquired.

Experiment Explanation: Spring's Modulus of Elasticity According to the Length

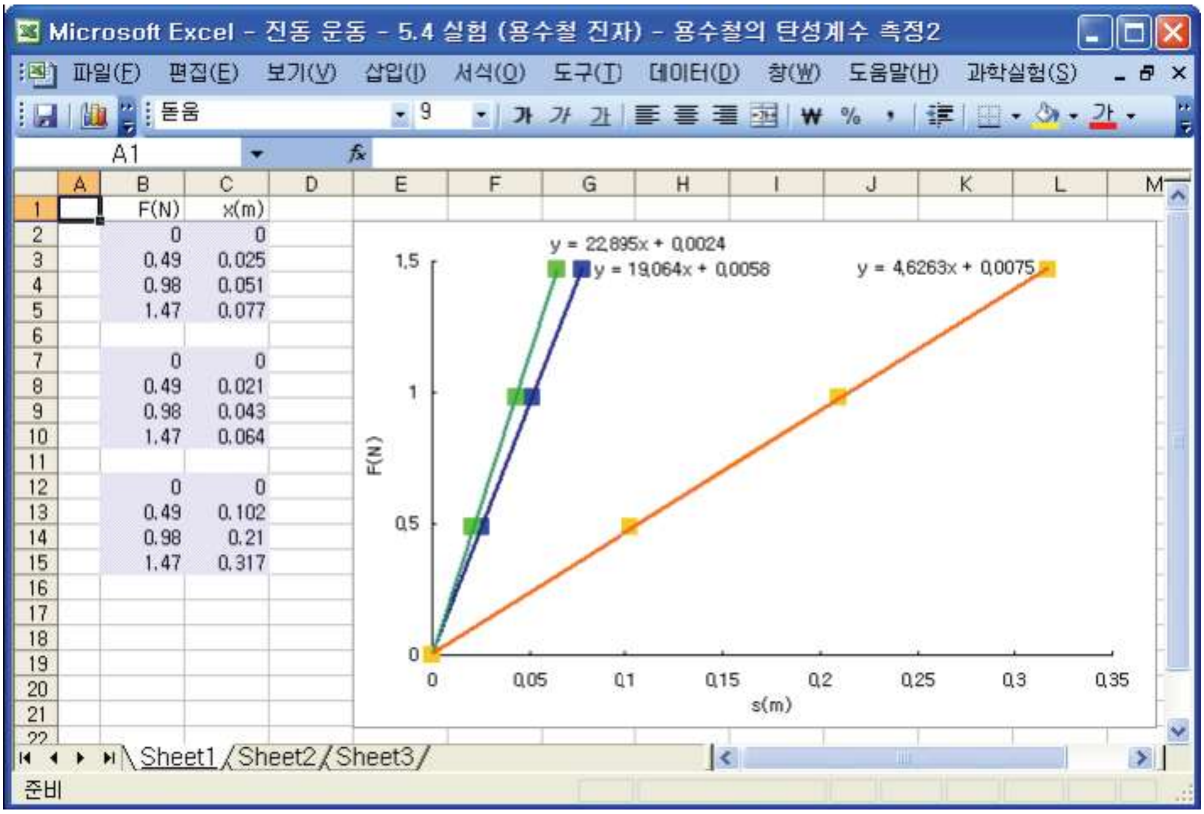
1. Fill out next table with the values of force and the expanded length.

[58] After drawing the graph of force ($F = kx$) and displacement (x) by increasing the number of the pendulums hung to the spring, calculate the gradient. Then it is the modulus of elasticity (k).

	Spring1	Spring2	Spring3
	____cm	____cm	____cm
Expanded Length(cm)	Force(N)	Force(N)	Force(N)
1			
2			
3			
4			

Table 5.4.1 experiment result: force and expanded length

2. Using table 5.4.1, draw a graph about the force and the expanded length as in picture 5.4.4. What is the modulus of elasticity calculated from the graph's gradient?



Picture 5.4.4 graph about the force and the expanded length (example)

Spring 1's modulus of elasticity __________________N/m

Spring 2's modulus of elasticity __________________N/m

Spring 3's modulus of elasticity __________________N/m

3. From this result, explain how the modulus of elasticity becomes different according to the lengths of springs.

5.4.3 Experiment B: Oscillation of a Spring Pendulum

Experiment Prediction: Period and Amplitude of Oscillation according to a Ball's Size and Mass

1. Oscillate a ball with a big radius and with a small radius.

   a. Which one has faster period of oscillation?

   b. Does the damping resistance of the air which the ball gets affect the period or amplitude of the oscillation?

2. If the damping resistances are similar, how will the amplitude and the period be different according to the mass hung to the spring?

Experiment Process

1. Measure the masses of big ball and small ball with the electronic scale and the circumference with the cm measuring tape.

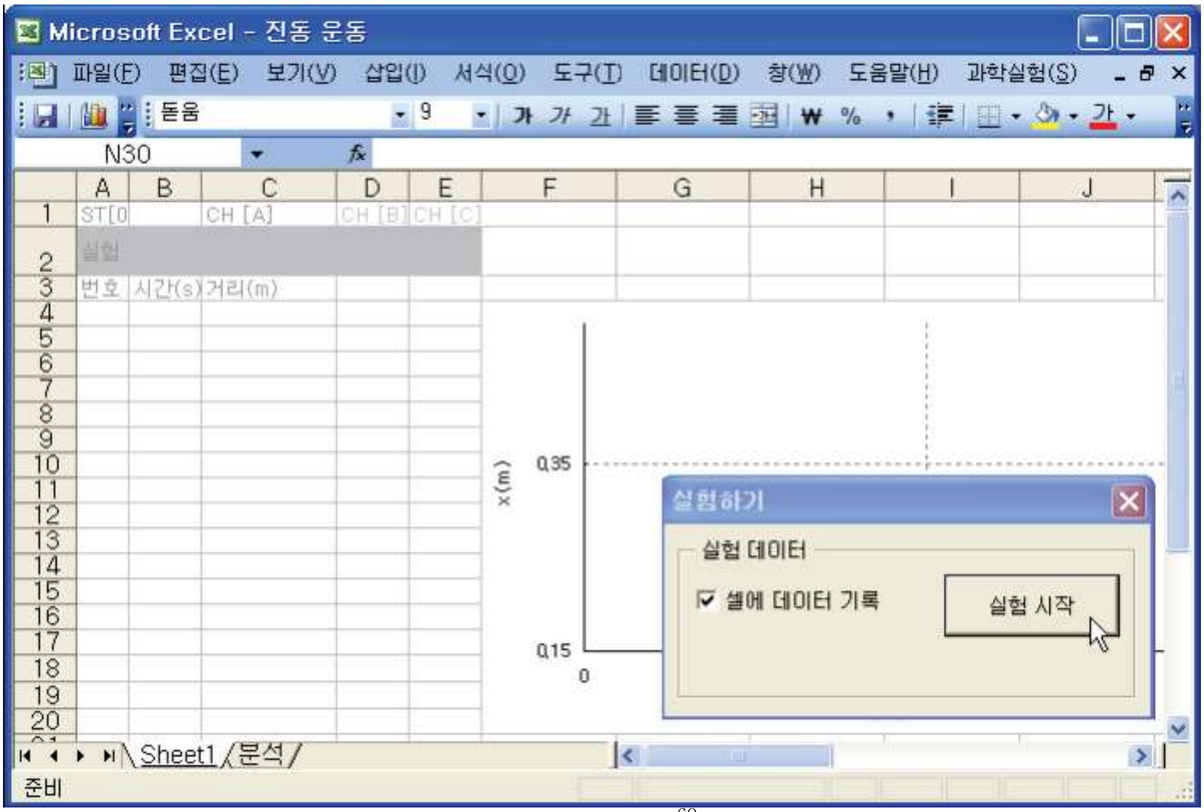
Picture 5.4.5 mass measuring of a big ball and a small ball[59]

2. As in picture 5.4.1, prepare for measuring the oscillation by hanging the ball to the spring.

   a. Put the motion sensor on the floor so that it can measure the location of the oscillating ball and connect the sensor and the computer.

   b. Open "Oscillation.xls" file.

[59] As an example, the big ball's mass is 113.7g, circumference is 130cm, and the small ball's mass is 280g, circumference is 43cm in this experiment. You can change the big ball's mass by injecting water.



Picture 5.4.6 experiment with "Sheet1" of Excel[60]: If you click [Start Experiment] button, data is collected within the sheet.

3. Open [Science Cube]-[Experiment Setting] window in worksheet menu and set the measuring interval as 0.05 second, and the experiment time as 60 second.

4. Open [Science Cube]-[Experiment] window in worksheet menu and click [Start Experiment] button. After that, the experimental data will be collected within the sheets[61].

5. As in picture 5.4.7, when the ball is in a state of equilibrium, lift it up lightly to a certain height. In this state, let the ball go lightly and oscillate it.

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[60] Use the prepared "Oscillation.xls" file. This file contains "Sheet1" and "Analysis" sheets. You can download this file at www.sciencecube.com.

[61] The supersonic wave perceiving part of the sensor should be placed under the ball. Move the ball up and down slowly so that you can check out whether the data reflects the ball's location correctly. If it is incorrect, move the motion sensor a little bit.

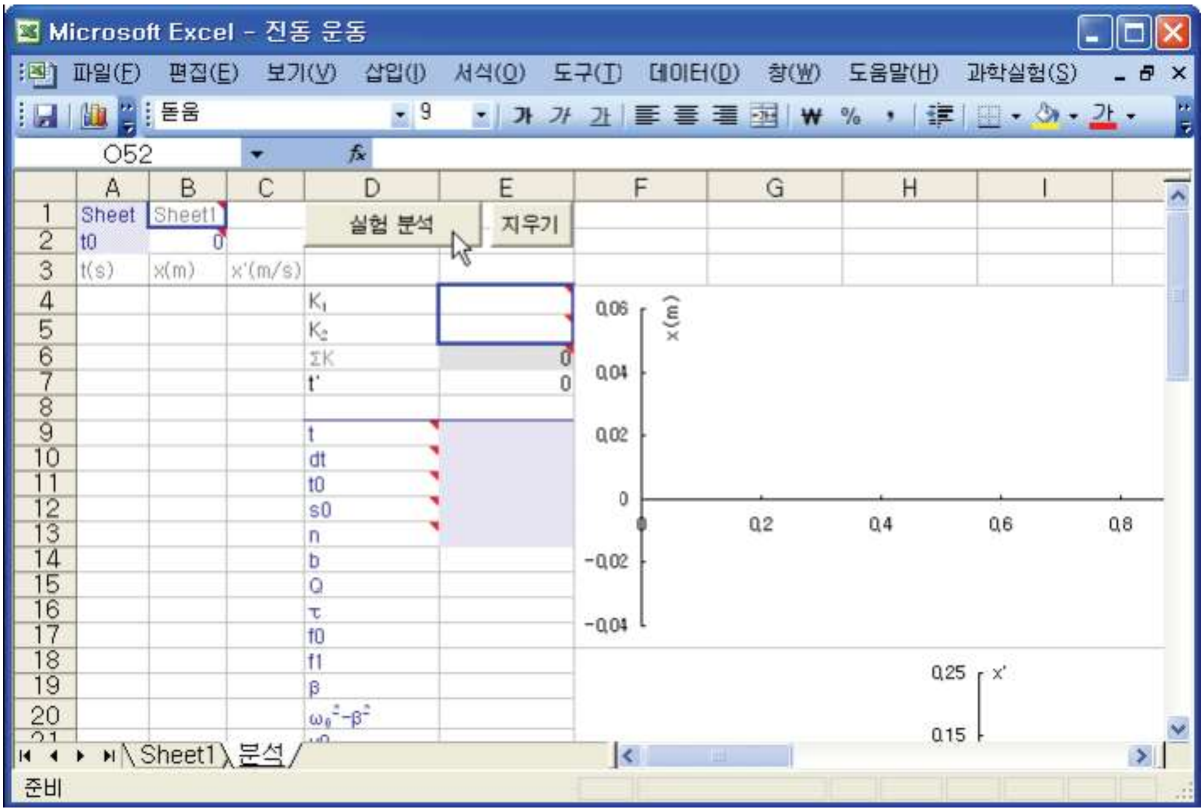
Picture 5.4.7 lifting up a ball to a certain height in the state of equilibrium

6. When the oscillation subsides, stop data collecting by clicking [Experiment]-[Stop Experiment] button and analyze the result in "Analysis" sheet.

a. As in picture 5.4.8, in "Analysis" sheet of "Oscillation.xls" file[62], input "Sheet1", the name of the sheet in which the data is collected, in cell B2, and record the modulus of elasticity in cell E.

b. If you click [Experiment Analysis] button in "Analysis" sheet, the data will be analyzed automatically and the results will be shown in "Analysis" sheet.

[62] This file contains "Sheet1" and "Analysis" sheets.



Picture 5.4.8 analyzing the results in “Analysis” sheet of “Oscillation.xls” file: If you click [Experiment Analysis] button, it will bring the experimental data of “Sheet1” automatically, analyze it, and record the results.

Deepened Experiment: Experiments with Various Physical Circumstances

1. Repeat the process above and execute the experience with springs that have different modulus of elasticity.

2. Execute the experiment to observe the period of the oscillation and the graph of damping amplitude according to the mass, size and shape of the object[63] hung to the spring.

[63] Consider the case in which the damping occurs in complicated conditions according to the object's composition and form.

3. Execute an oscillation experiment as in picture 5.4.9. Connect 50g pendulum to two springs and oscillate it.

a. Measure the pendulum's oscillating period using photogate[64].

b. Remove one spring under the pendulum and oscillate it to compare the result.

Picture 5.4.9 oscillation with two springs: Oscillate the pendulum up and down so that it blocks and unblocks the photogate.

4. In picture 5.4.9, use the motion sensor instead of photogate[65] to measure the displacement of the system and draw x-t graph.

[64] When photogate is connected to the computer and Excel workbook is open, it will be set up automatically as strobo timing mode. In this mode, the motion period of a object can be measured. Strobo timing mode is the general way that photogate measures the period of "unblocked-blocked-open" with the moving object. Besides this, there is such way as gate timing, which measures "open-blocked".

[65] Because photogate cannot measure the displacement of the system, motion sensor should be used. Find the way that the motion sensor can perceive the pendulum's displacement and execute the experiment design.

Experiment Explanation: Characteristics of a Spring Pendulum's Oscillation

1. Make the analysis result as a table.

	Experiment 1	Experiment2	Experiment3
b			
Q			
τ			
f_0			
f_1			
β			
$\omega_0^2 - \beta^2$			
m'			
Modulus of Elasticity(K)			
Ball's Volume			
Ball's Mass			

Table 5.4.2 results of a big ball and a small ball's oscillation

2. Explain the result in table 5.4.2.

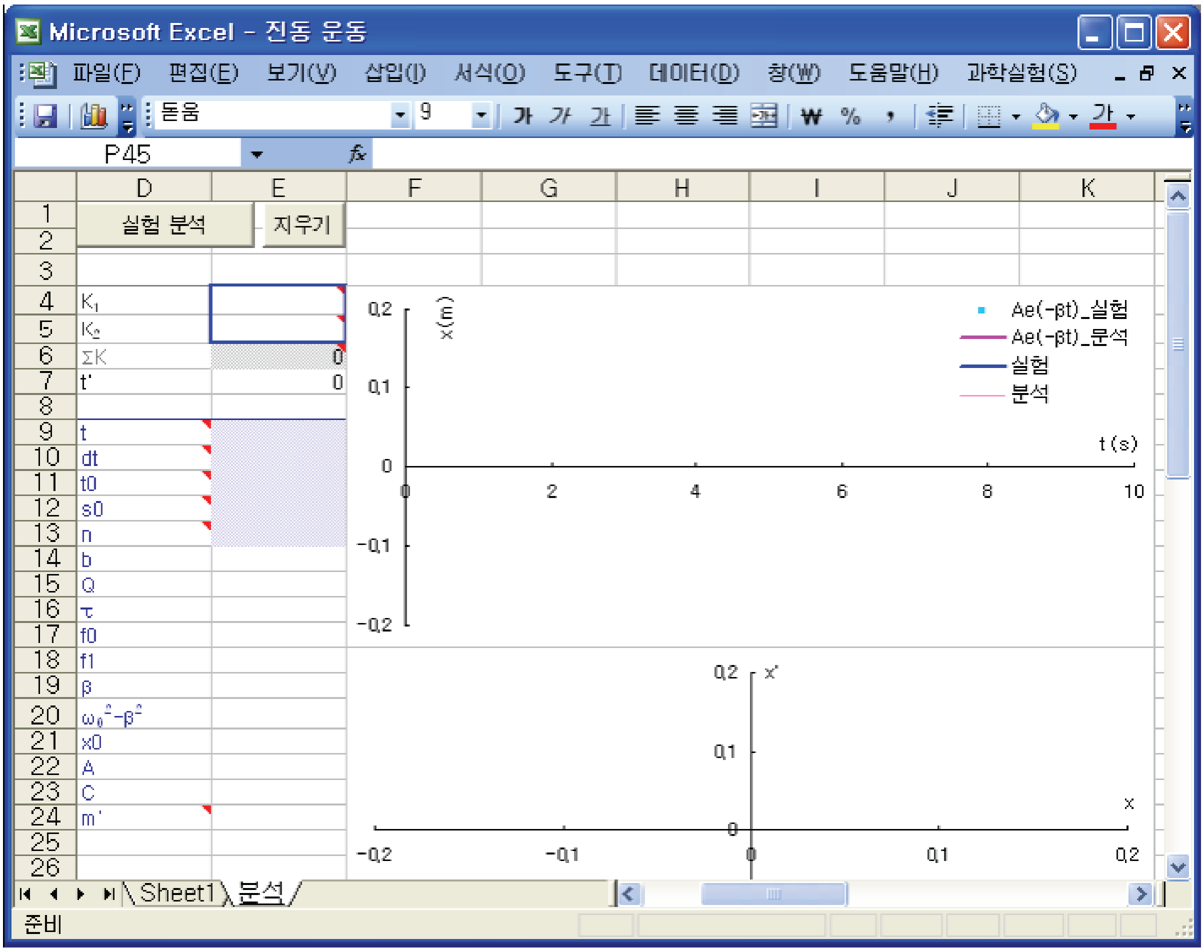
   a. What does the period of a spring pendulum is related to?

   b. What is the reason of a spring pendulum's delicate damping? Compare it with the result of free oscillation.

3. Explain the graph of displacement and time(x I t) and the graph of phase

   space( )[66]

[66] "Analysis" sheet contains x I t , graph charts. If you click [Experiment Analysis] button, the analyzed data will be recorded automatically in column A,B and C, and graphs will



Picture 5.4.10 analyzing the graph of displacement and phase space: If you click [Experiment Analysis] button in “Analysis”, the analyzed data of time, displacement, and velocity will be recorded automatically in column A, B and C. And x I t , graphs prepared in “Analysis” sheet will be drawn automatically.

4. Draw the energy relationship graph of the system[67] and explain.

---

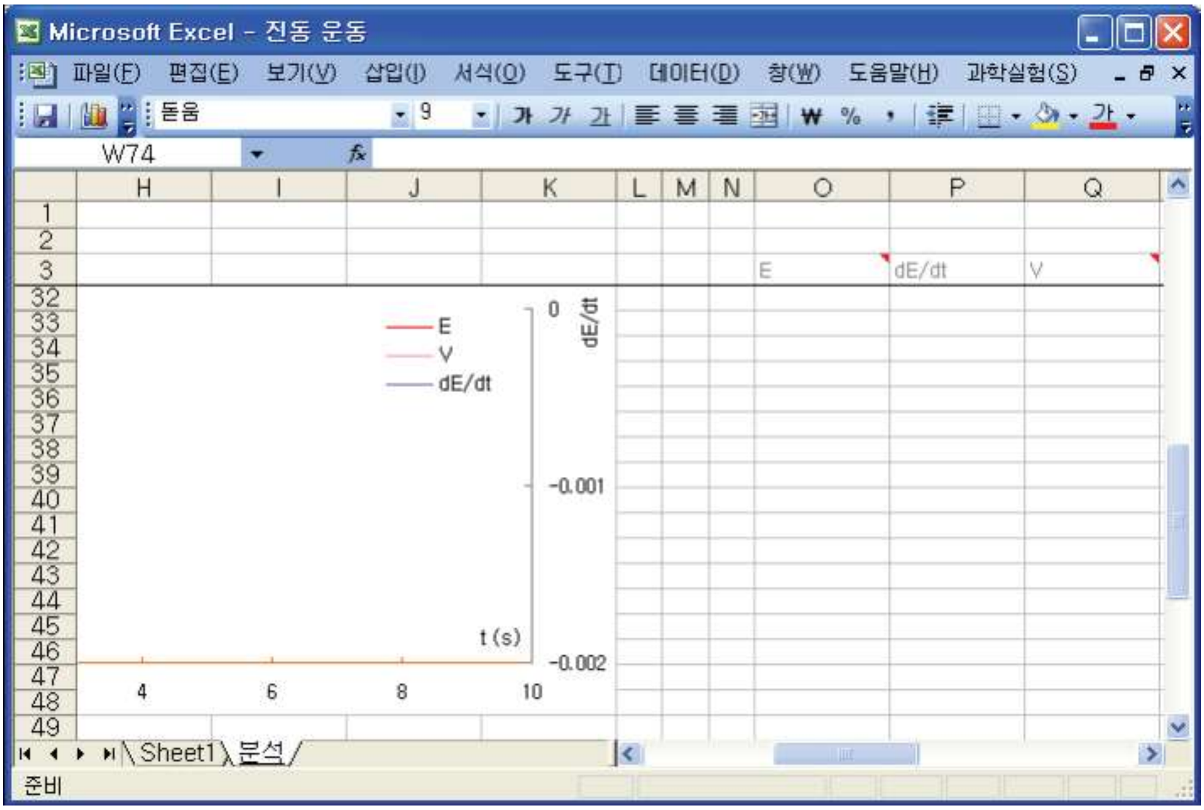
be drawn in the chart.

[67] If you click [Experiment Analysis] button in “Analysis” sheet, the data of total energy(E), energy loss rate(dE/dt), potential energy(V) is recorded in column O,P and Q, and then graphs are drawn in the prepared E-t, dE/dt),V-t charts. According to the experiment conditions, the scale values of x axis (time) and y axis (energy) can be very different. To draw graphs properly in the chart area, change the maximum and minimum values in [Axis Form]-[Scale] and draw them.

a. Graph of total energy and time(E-t)

b. Graph of energy loss rate(dE/dt-t)

c. Graph of potential energy and time(V-t)



Picture 5.4.11 analyzing the graph of energy (E) and time (t): if you click [Experiment Analysis] in "Analysis" sheet, the data will be recorded in column O, P and Q.

5. When the ball hung to the spring oscillates as in the free oscillation, is free frequency almost same as damping frequency? Explain this with the result.

a. Free Frequency ______________________

b. Damping Frequency ______________________

Deepened Explanation:

1. As in picture 5.4.9, explain how the two spring system's period of oscillation is different from one spring system's period of oscillation with the experiment result.

5.4.4 Experiment Questions

1. Explain how the damping can be different according to the relationship between the volume and mass of the ball hung to the spring.

   a. when the volume is consistent and the mass is different

   b. when the mass is consistent and the volume is different

   c. How is the air resistance related to the damping resistance?

2. As in picture 5.4.7, how does the height of the ball[68] affect the result?

3. In the cases below, explain how the period and amplitude of the spring pendulum's oscillation changes.

   a. when the same objects are hung to the springs that have different modulus of elasticity

   b. when objects that are same in shape and size and different in mass are hung to the same springs

4. Explain the displacement, velocity and acceleration of the ball hung to the spring.

   a. the velocity and acceleration when the displacement of the ball is at the maximum toward + or − direction

   b. the velocity and acceleration when the displacement of the ball is 0

[68] This is the maximum displacement of the ball when it starts oscillating.

Deepened Questions

1. In this experiment, execute the experiment analysis by measuring not the spring's modulus of elasticity but the mass of the ball[69]. How will the result be different? Explain the differences. Using the result, calculate the spring's modulus of elasticity.

2. Set the equation of motion[70] for the two spring system and using the experimental data, calculate the solution for the frequency and displacement of the system in the way of data analysis based on physical modeling and solution finding.

3. Analyze and explain the factors that affect the ball's oscillation.

a. factual evidences analyzed through the experiment

b. physical values that can be explained through the evidences

c. various cases of oscillation predictable based on the experiment conditions

[69] To do this, you should modify VBA code in "Oscillation.xls" file. VBA original code can be modified by opening VBE window.
[70] When the effect of the spring's mass is neglected, the two spring's total modulus of elasticity can be expressed as $k = k_1 + k_2$ and the solution for exercise 5.2.1 can be applied.

5.5.

Experiment: Damped Oscillation

5.5.1. Experiment Outline

With an oscillation experiment of a mass-spring system, let's calculate the location, velocity and energy of a mass according to time and analyze the frequency and damping constant so that we can understand what causes the damping of the system. By this, you can understand physical concepts according to the physical circumstances of the damped oscillation.

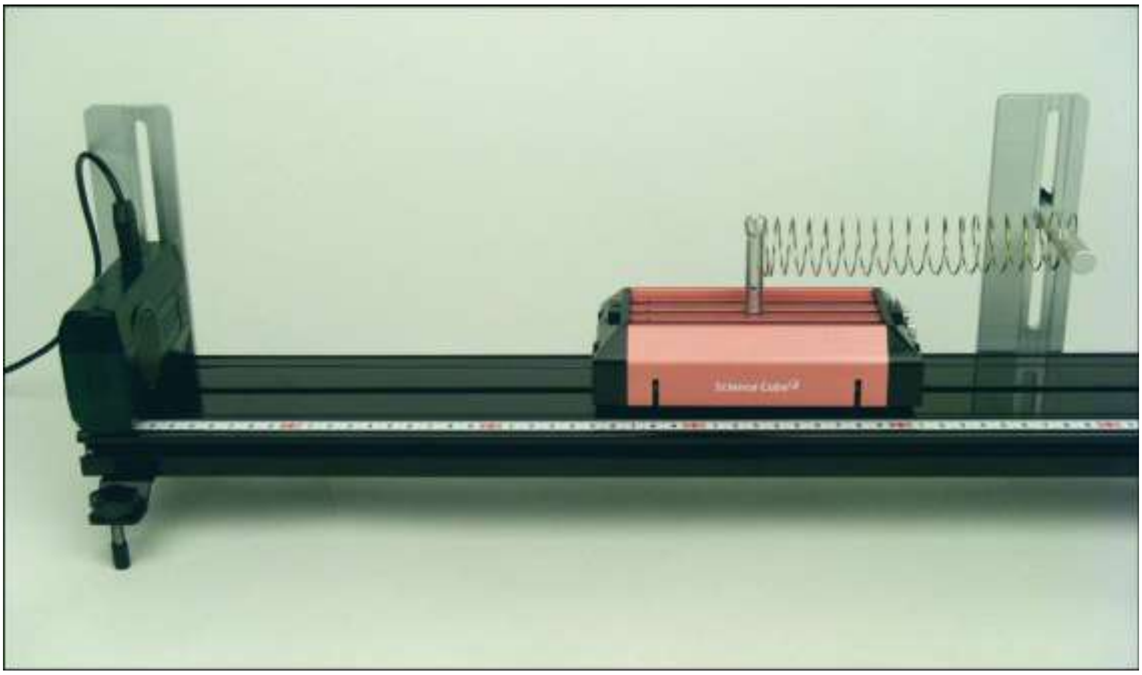
Goal

Understanding characteristics of a mass-spring system's damped oscillation

Required Equipments

Motion sensor	1
Spring (pushing 1, pulling 2)	
Cart	1
Track	1
Pendulums (50g)	1

5.5.2 Experiment A: Oscillation of a Mass-Spring System (1)



Picture 5.5.1 oscillation experiment of a mass-spring system: with the motion sensor, measure the oscillation displacement of a cart on the horizontal track.

Experiment Prediction: Prediction of Experimental Evidences[71] about the Causes of Damped Oscillation

1. As in picture 5.5.1, consider the case when the air resistance is neglected in the pushing-pulling spring- cart system's oscillation.

a. What is the biggest cause of damping resistance?

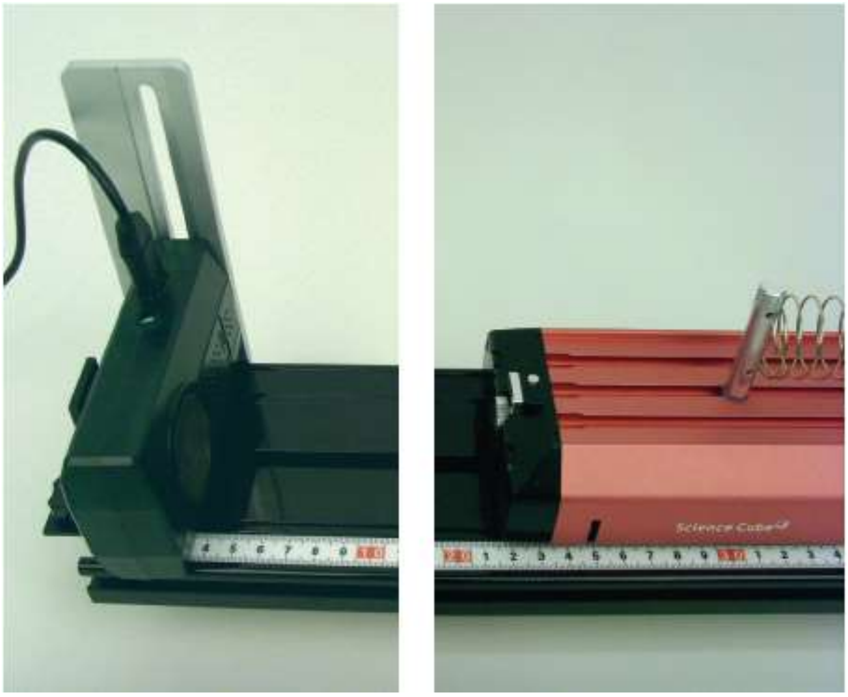
b. How can it be proved by experiment?

c. What makes the oscillation period different?

d. How will the cart's displacement change according to time?

[71] Based on the things from the theories, predict the experiment's evidences in advance and understand them with the real experiment.

Experiment Process:



Picture 5.5.2 preparation of motion sensor and cart[72]

1. As in picture 5.5.2, put the cart on the track and set up the spring to the cart.

2. As in picture 5.5.2's left one, set up the motion sensor upon the track so that it can measure the cart's location.

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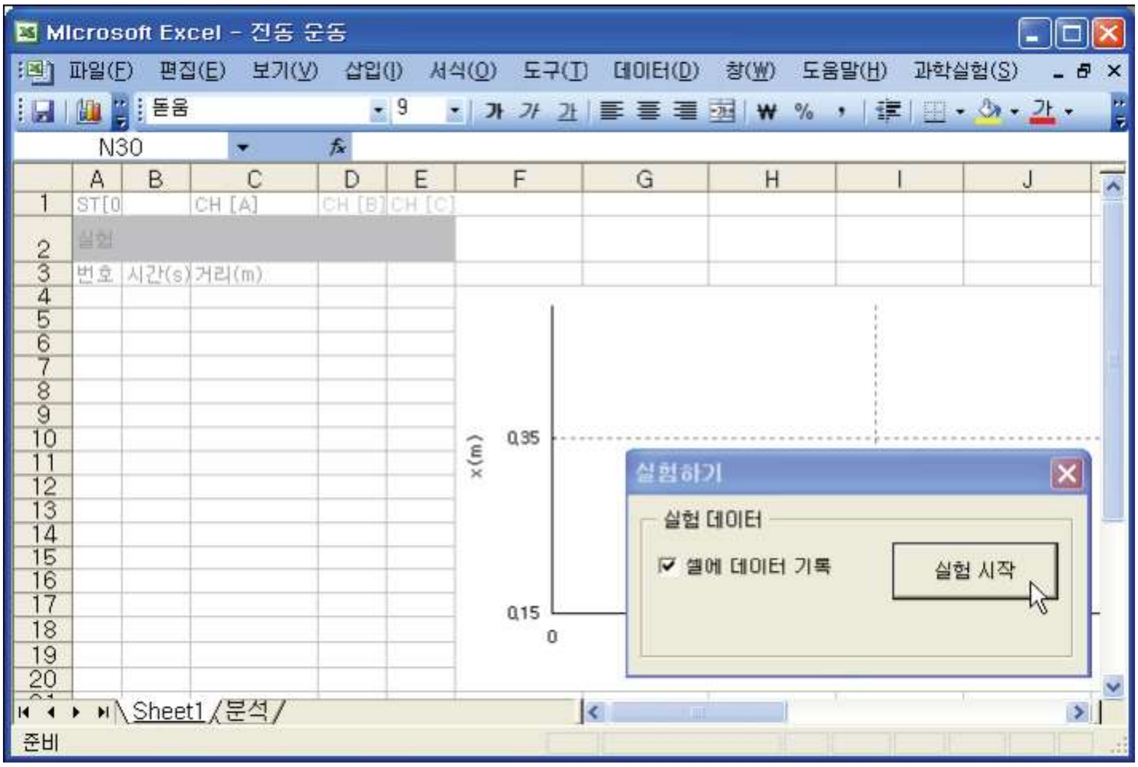
[72] The cart's wheel has minute bearing construction and the area that is contacted with track is constructed narrowly as the blade so it can reduce the friction resistance caused by the contacted area.

3. Connect the sensor and the computer and open “Oscillation.xls” file.

a. Open [Science Cube]-[Experiment Setting] window in worksheet menu and set up the measuring interval as 0.05 second and the experiment time as 60 seconds.



Picture 5.5.3 experiment with “Sheet1” of Excel: If you click [Start Experiment] button, data will be collected within the sheet.

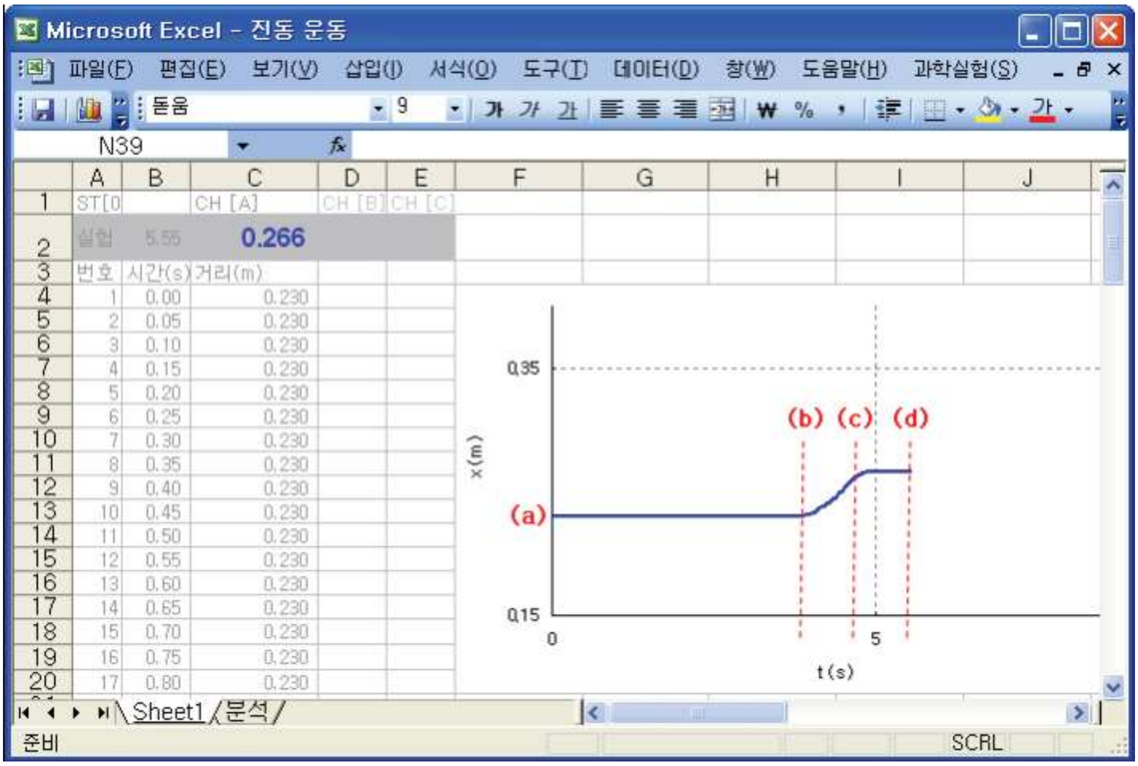
b. Open [Science Cube]-[Experiment] window in worksheet menu and click [Start Experiment] button. If you click the button, as in the graph section (a)-(b) of picture 5.5.4, the experimental data will be collected in the sheet[73].

c. As in picture 5.5.1, in the state of equilibrium, move the cart slowly close to or far from the motion sensor[74]. As in the graph section (b)-(c) of picture 5.5.4, data will be collected.

[73] Set up the supersonic wave perceiving part of the motion sensor toward the cart. Move the motion sensor to check out whether the data reflects the cart’s location accurately. If the measured value is inaccurate, move the motion sensor little by little.

[74] The moving distance will be the maximum value of the oscillation’s amplitude.

d. In the section (b)-(c), in which the cart has been moved, release the cart gently and make it oscillate.



Picture 5.5.4 starting experiment in Excel workbook: If you release the cart gently at (d), it will start the oscillation.

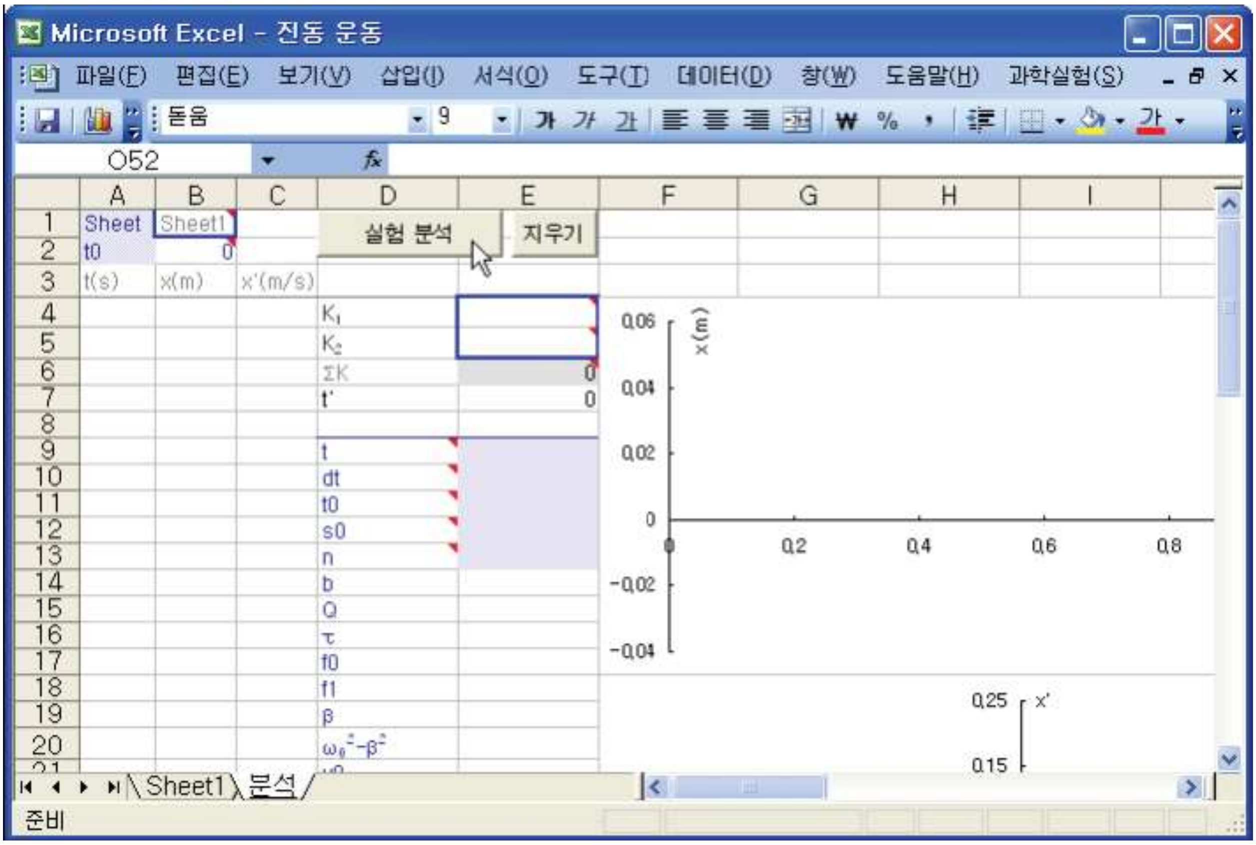
4. When the oscillation fades, click [Stop Experiment] in [Experiment] window and stop data collecting.

5. As in picture 5.5.5, in the “Analysis” sheet of “Oscillation.xls” file[75], write the data collected sheet’s name “Sheet1” into cell B2 and write the modulus of elasticity in cell E4.

a. If you click [Experiment Analysis] button, the data will be analyzed automatically, and the results will be shown in “Analysis” sheet.

[75] This file has been explained in the experiment of a spring pendulum and it can be used here, too. This file contains “Sheet1” and “Analysis” sheet.

b. Explain and discuss the results analyzed in the cell area from E14 to E 24 of “Analysis” sheet.



Picture 5.5.5 analyzing the results in “Analysis” sheet of “Oscillation” sheet

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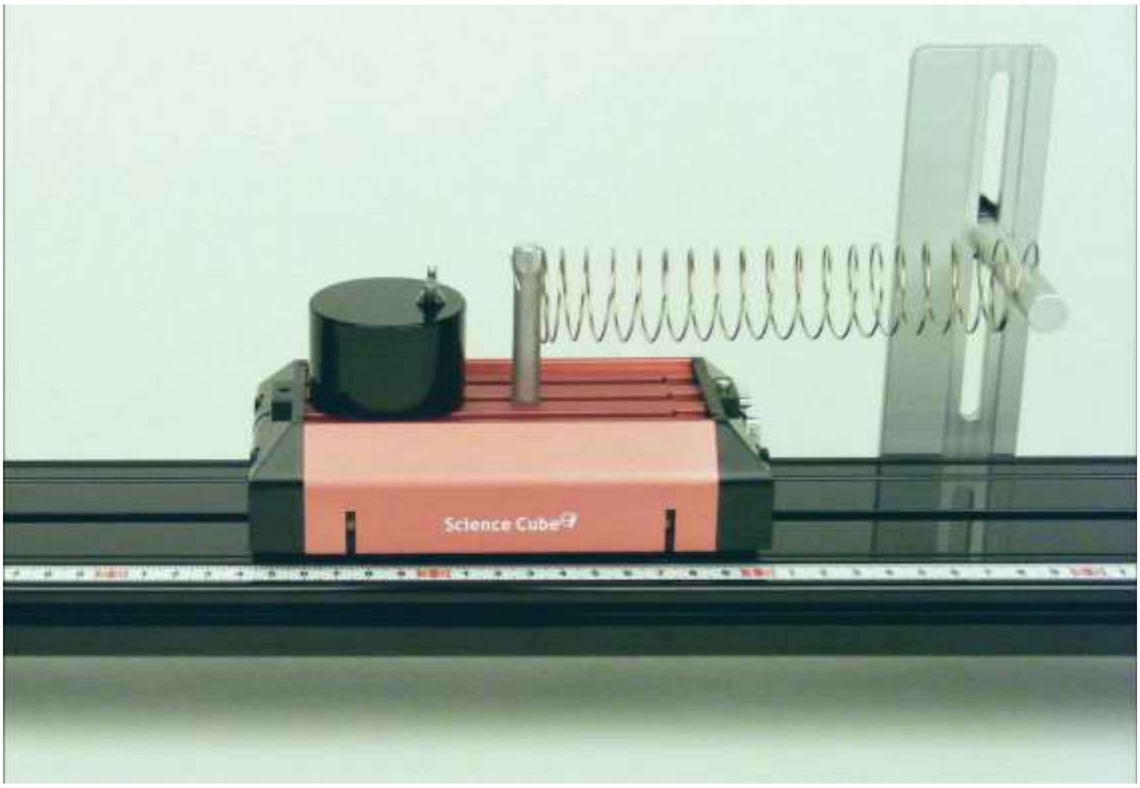
Deepened Experiment: Experiments with Various Physical Circumstances

1. Repeat the process above and execute the experience with springs that have different modulus of elasticity.

2. As in picture 5.5.6, change the cart’s mass and execute the experiment to observe how the period of oscillation and the amplitude curve will change.



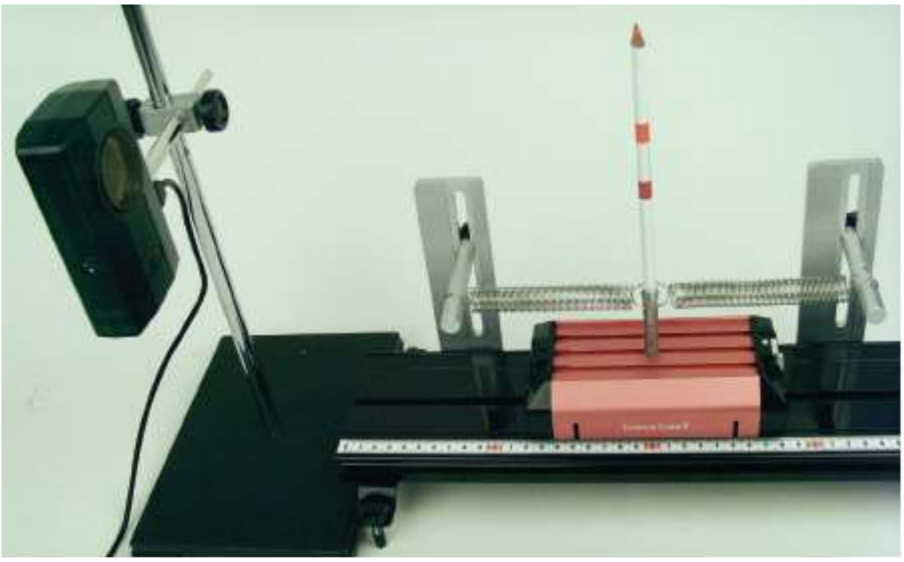
Picture 5.5.6 experiment with different masses: the circumstance that the cart is heavy because a 500g mass is put onto the cart

3. Picture 5.5.7 shows the oscillation of a cart with two pushing-pulling springs. Execute an experiment to compare with the oscillation with one spring.

a. Set up a 15~20cm length poll to the cart so that the motion sensor can perceive it. By executing the preliminary experiment, set up the motion sensor on a stand at the best height of perceiving the cart[76].

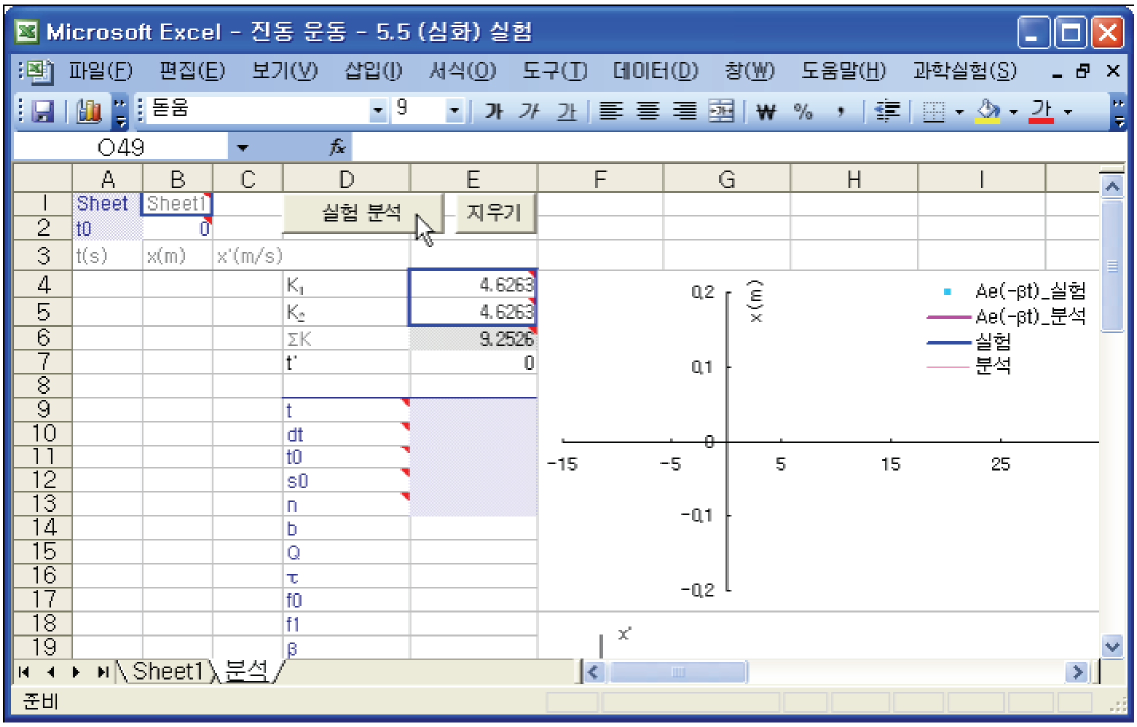
b. Execute the experiment in “Sheet1” of “Oscillation.xls” file and analyze the result n “Analysis” sheet.

[76] Motion sensor operates in this way: it sends diffusing supersonic waves within 15° range and perceives the signals reflecting from objects. In the circumstance in picture 5.5.7, it is difficult to measure the cart's location accurately because the spring which is on the left is so close to the motion sensor that it interferes. Therefore, you should set up the sensor higher and set up a perceiving pole onto the cart.



Picture 5.5.7 oscillation of a mass-spring system which consists of two springs

c. As the oscillation fades, stop data collecting and input the modulus of elasticity K_1, K_2 and analyze the results.



Picture 5.5.8 experiment analysis of a system which consists of two springs: Input the modulus of elasticity K_1 in cell E4, K_2 in E5 and click [Experiment Analysis] button.

Experiment Explanation: Damped Oscillation

1. Fill out next table with the results of experiment analysis.

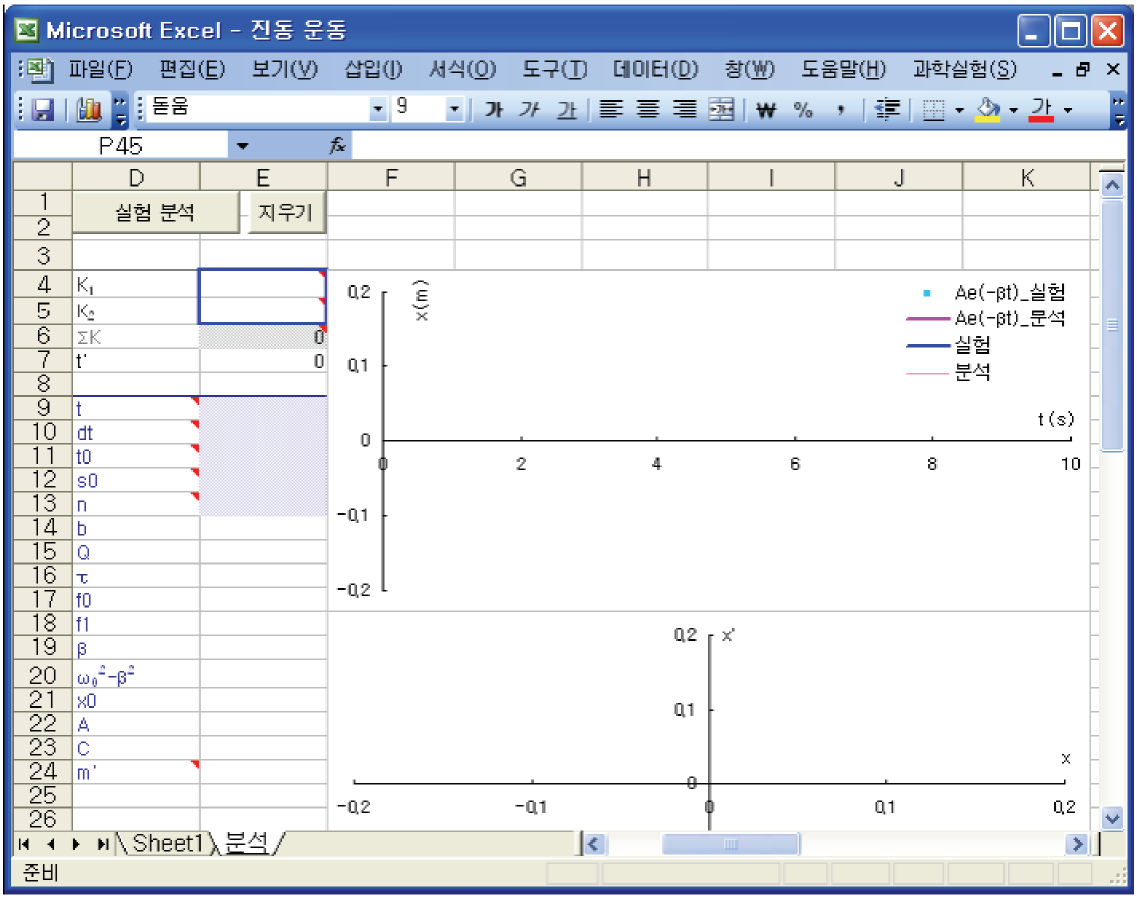
	Experiment 1	Experiment 2	Experiment 3
b			
Q			
τ			
f_0			
f_1			
β			
$\omega_0^2 - \beta^2$			
m'			
Modulus of Elasticity(K)			
Cart's mass			

Table 5.5.1 result of cart's oscillation

2. Explain the results in table 5.5.1. What is the factor that causes periodical damping to the cart's motion?

3. Analyze the graph of displacement and time($x-t$), displacement and phase space($x-\dot{x}$)[77] and explain them.

[77] "Oscillation.xls" file has been explained in the experiment of a spring pendulum and it can be used here, too. "Analysis" sheet contains $x-t$, $x-\dot{x}$ graphs. If you click [Experiment Analysis] button, the result analysis data will be recorded automatically in column A, B and C, and the graphs will be drawn in the chart.



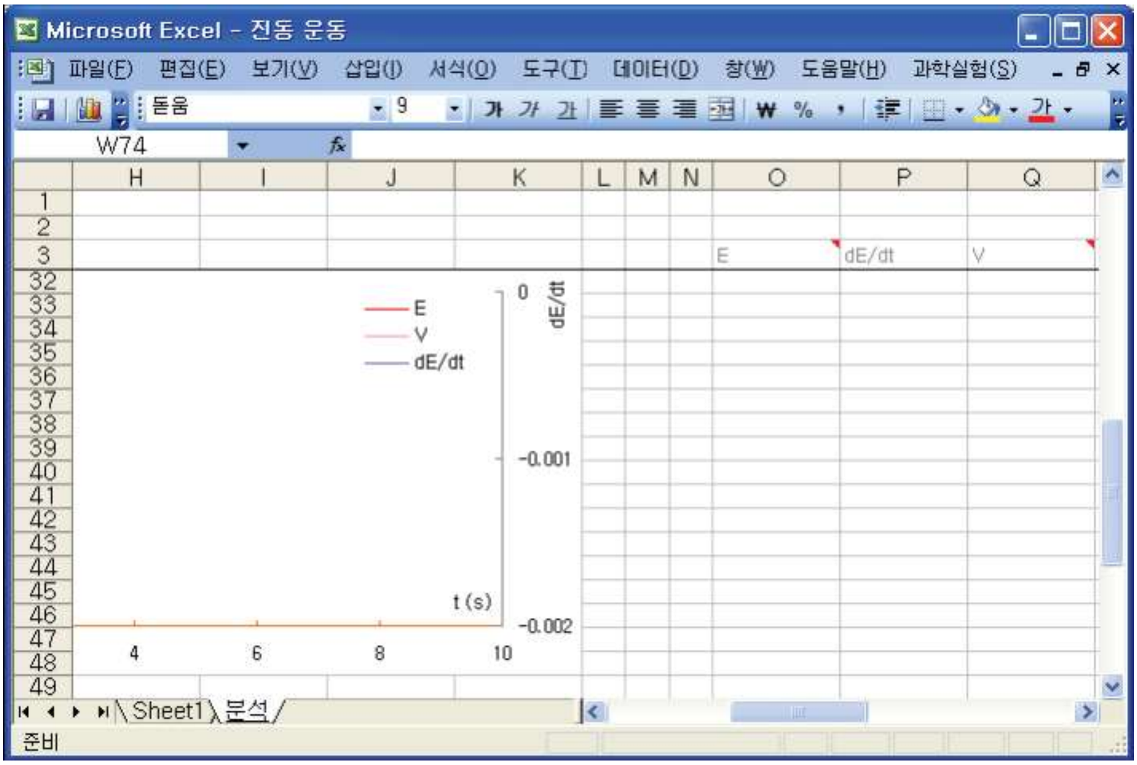
Picture 5.5.9 analyzing graphs of displacement, phase space: If you click [Experiment Analysis] button in "Analysis" sheet, the analyzed data of time, displacement, and velocity will be recorded in column A, B, and C. And then, the prepared $(x-t)$, $(x-\dot{x})$ graphs will be drawn automatically.

3. Are the cart's natural frequency and damping frequency different greatly? Or aren't they? Explain it with the experiment results.

4. How are the time constantγ, damping constant β different according to time? For example, how are they different when the cart is heavy?

5. Analyze graphs of total energy and time(E-t), energy loss rate and time(dE/dt-t) and potential energy and time(V-t)[78] and explain them.

[78] Refer to "Analysis" sheet of "Oscillation.xls" file, which has been explained already.



Picture 5.5.10 analyzing the graph of energy (E) and time (t): If you click [Experiment Analysis] button in “Analysis” sheet, the energy analysis data will be recorded in column O, P and Q.

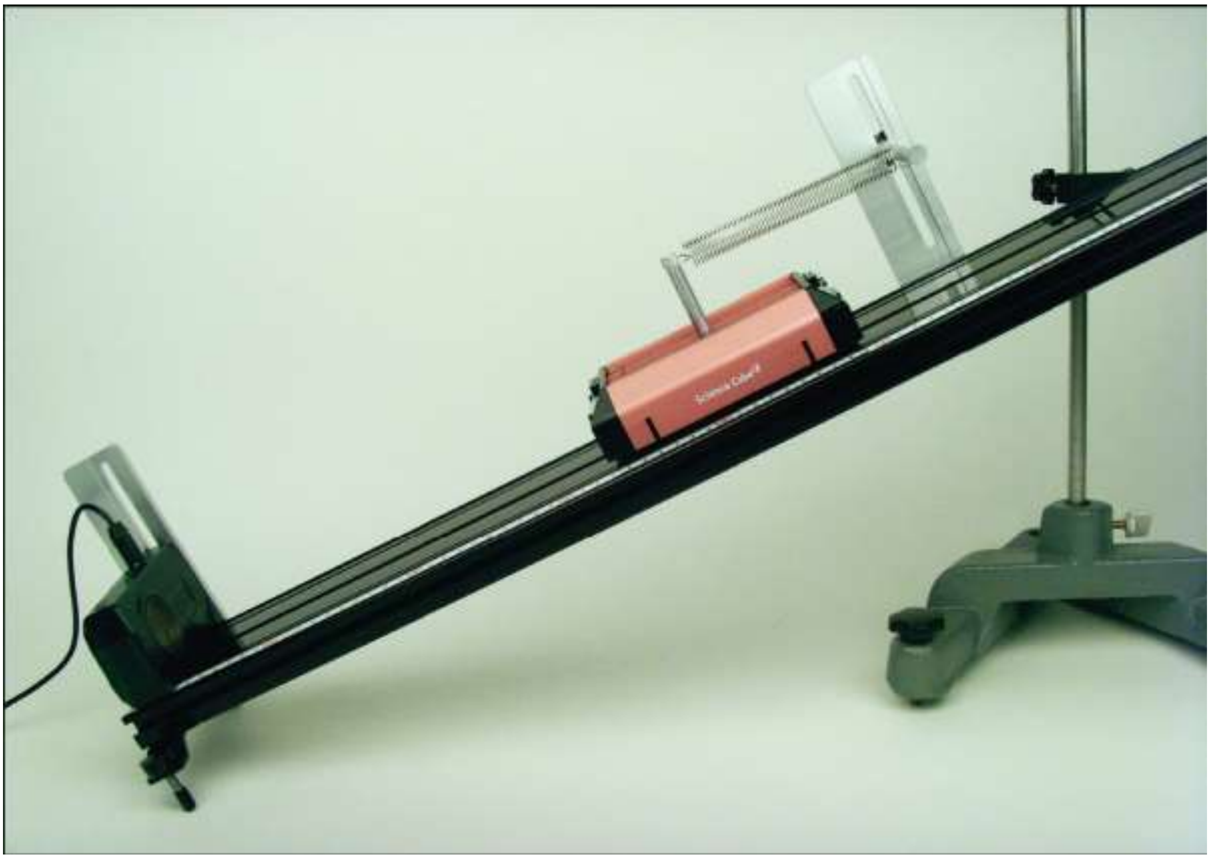
6. Is the cart shows the free oscillation as the oscillation of a spring pendulum? If not, what is the factor that affects the cart’s damped oscillation? Explain it with the experiment results.

Deepened Explanation:

1. As in picture 5.5.7, how is the oscillation period of a system with two springs different from that of a system with one spring?

2. In the circumstance of picture 5.5.7, If both ends of two springs are expanded and the length of springs gets even longer, how is the experiment result different?

5.5.3 Experiment B: Oscillation of a Mass-Spring System (2)



Picture 5.5.11 oscillation experiment of a mass-spring experiment: Measure the cart's displacement on the track with the motion sensor.

Experiment Prediction: Factors that Cause the Oscillation of a Pulling Spring

1. As in picture 5.5.11, predict what is related to the oscillation of the system with a pulling spring and a cart[79].

a. What is the reason that the expanded length of a spring in the state of

[79] In picture 5.5.1, if you use a pulling spring instead of a pushing-pulling spring, you cannot cause the oscillation. But, in case of the slope, you can cause the oscillation with a pulling spring.

equilibrium changes according to the gradient of the slope?

b. Concerned with the slope's gradient, let's consider whether the forces that are related to the cart's oscillation affect the period of oscillation.

Experiment Process:

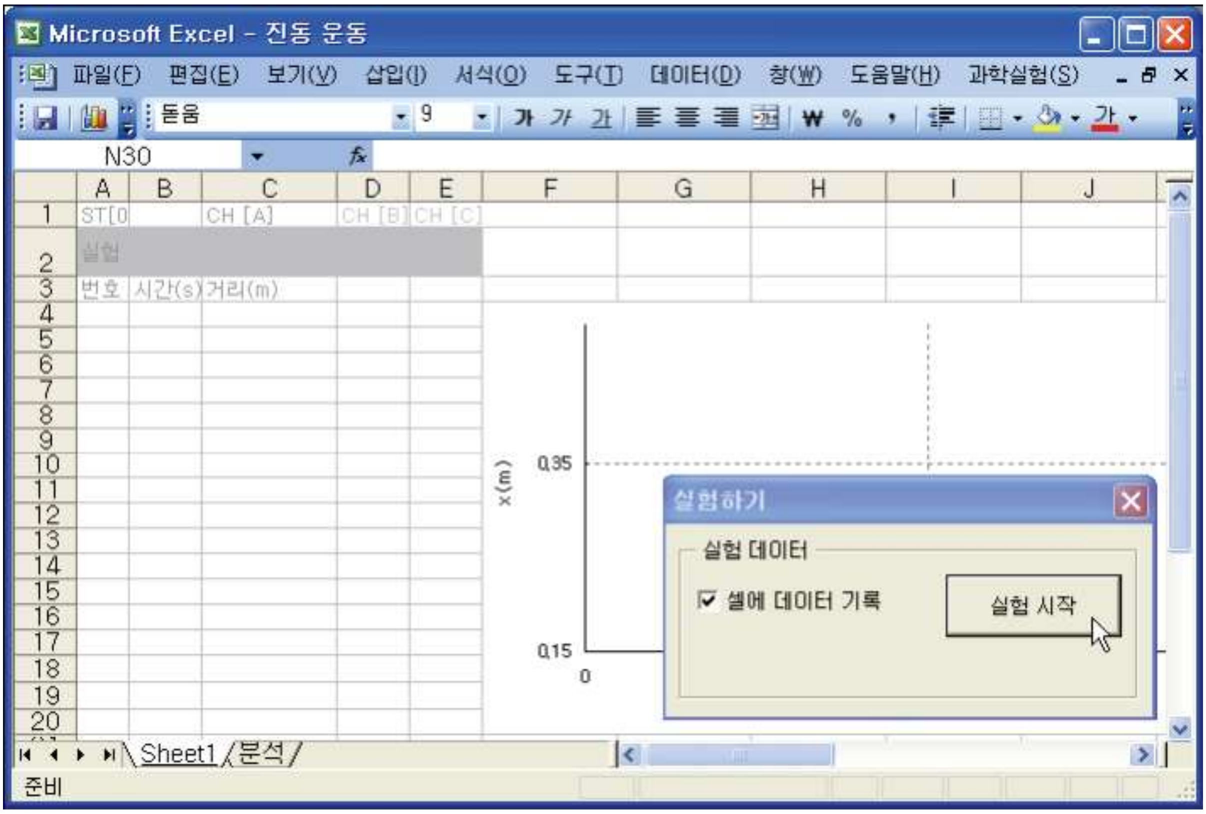
1. As in picture 5.5.1, tilt the track in a certain angle and make it as a slope.

2. AS n picture 5.5.11, put a cart on the track, set up a spring and set up the motion sensor on the track to measure the cart's location.

3. Connect the sensor with the computer and open the "Oscillation.xls" file.

a. Open [Science Cube]-[Experiment Setting] window in worksheet menu and set up the measuring interval as 0.05 second and the experiment time as 60 seconds.

b. As in picture 5.5.12, open [Science Cube]-[Experiment Setting] window in worksheet menu and click [Start Experiment] button.



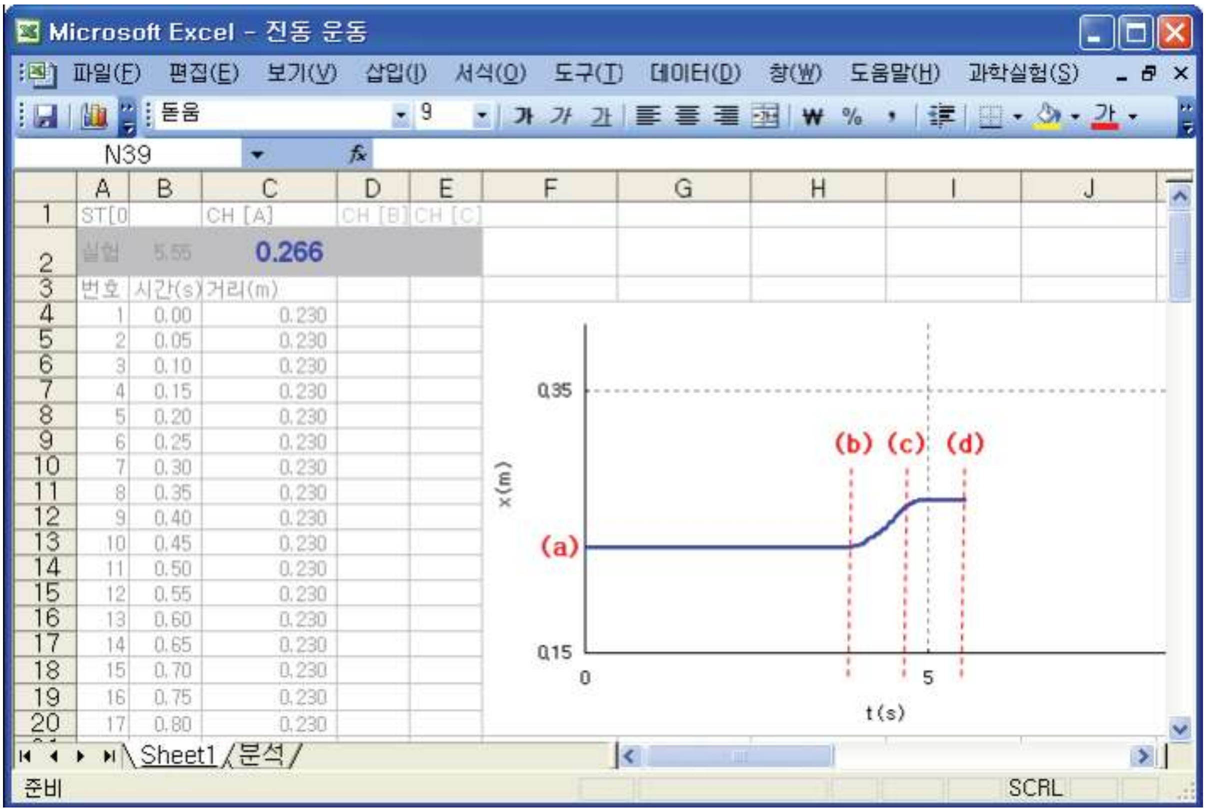
Picture 5.5.12 experiment with “Sheet1” of Excel: If you click [Start Experiment] button, the data will be collected into the sheet.

c. If you click [Start Experiment] button, as in the graph section (a)-(b) of picture 5.5.13, the experimental data will be collected into the sheet.

d. From the state of equilibrium, move the cart slowly to the upper part of the slope. The data will be collected as in the graph section (b)-(c) of picture 5.5.13.

e. From the (c)-(d) section[80] of picture 5.5.13., which is the state that the cart has been moved, release the cart gently and make it oscillate.

[80] the state that the cart has been stopped because of holding it after moving it from the state of equilibrium



Picture 5.5.13 starting experiment in Excel workbook: If you release the cart gently at (d), it will start oscillating.

4. As the oscillation fades, stop collecting data by clicking [Stop Experiment] of [Science Cube].

5. As in picture 5.5.14, in "Analysis" sheet of "Oscillation.xls" file[81], input "Sheet1 in

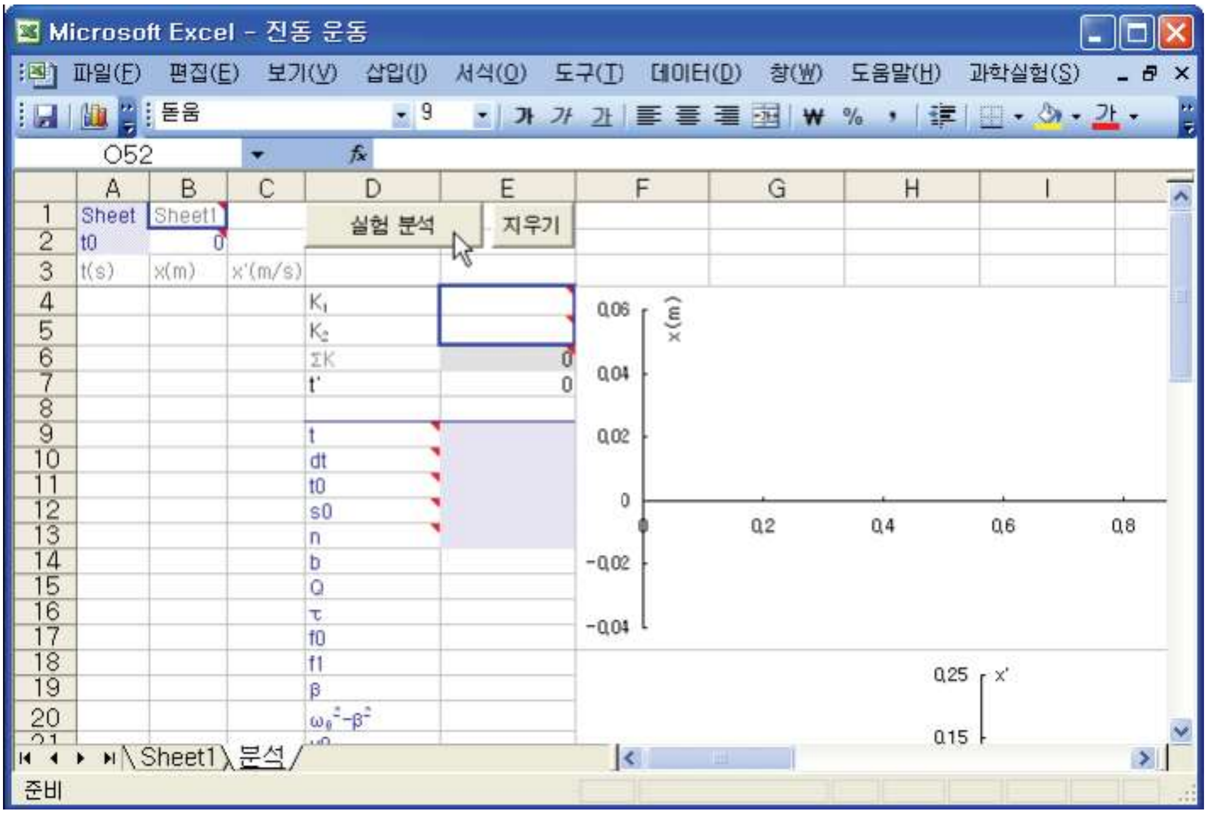
---

[81] This file has been used before in the Experiment A", which was the motion of a spring pendulum and the cart's oscillation on the horizontal plane. This file can be used

cell B2, which is the name of the sheet where the experimental data was collected, and write the modulus of elasticity in cell E4.

a. If you click [Experiment Analysis], the data will be analyzed automatically and the results will be shown in "Analysis" sheet.

b. Explain and discuss the analyzed results from cell area E4 to E24.

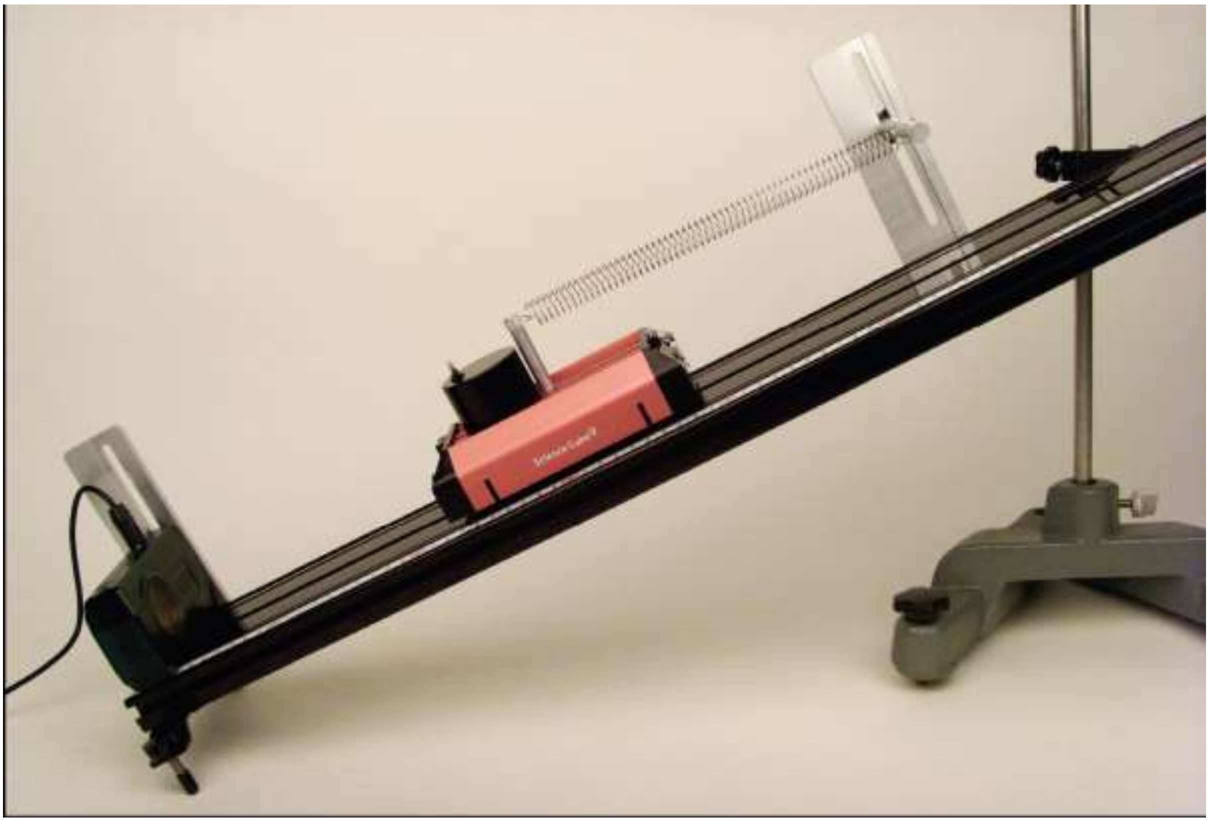


Picture 5.5.14 analyzing the results in "analysis sheet of "Oscillation.xls" file

generally in the damped oscillation of a mass-spring system.

Deepened Experiment: Experiments according to the Various Physical Circumstance

1. As in picture 5.5.15, change the cart's mass and observe how the period of oscillation and the shape of damping amplitude curve change. Execute the experiment to observe how the cart's mass affects the oscillation on the slope and compare this with the motion of a spring pendulum.



Picture 5.5.15 experiment with different mass: The cart is made heavy with a 500 g

mass on it.

2. As in picture 5.5.16, execute the experiment with big gradient of the slope[82]. You should see how the slope's gradient affects the result and execute the experiment to observe how the period of oscillation and the shape of damping amplitude curve change.

a. Connect the spring low but not touching the surface of the track[83].

b. As the explanation of picture 5.5.3, execute the experiment in "Sheet1" of "Oscillation.xls." file and analyze the results in "Analysis" sheet.

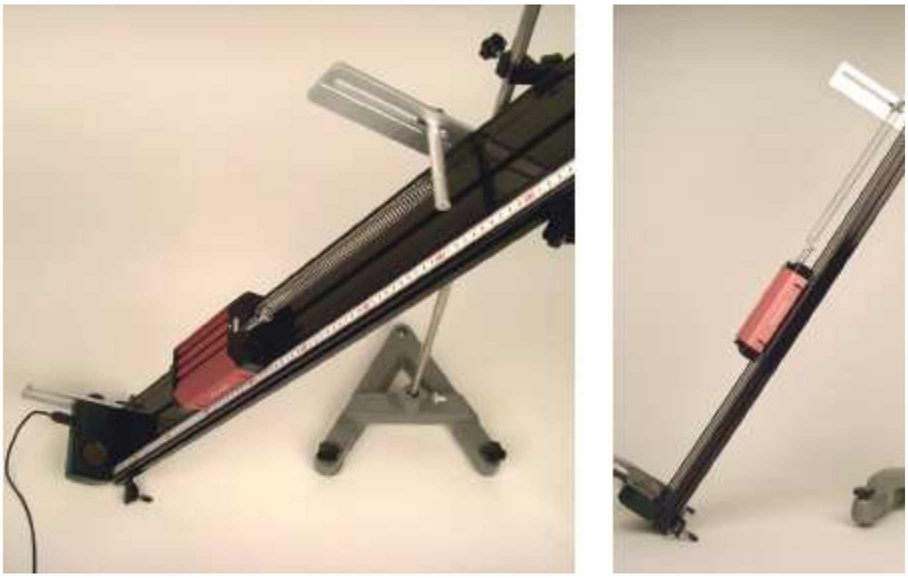
c. Get other results by changing the slope's gradient and analyze them.

d. Compare the results with the experiment of oscillation on the horizontal surface and explain them.

---

[82] Give attention to the experiment security. Note that the cart should not fall from the track to the ground or crash into the motion sensor.

[83] If the spring is connected high above the track, the cart will get loose when the slope's gradient is big.



Picture 5.5.16 experiment with bit gradient[84]

3. In the circumstance same as picture 5.5.17, make the slope's gradient constant, connect a spring with different modulus of elasticity to the cart and execute the experiment.

a. Connect springs that have different modulus of elasticity and lengths to the cart.

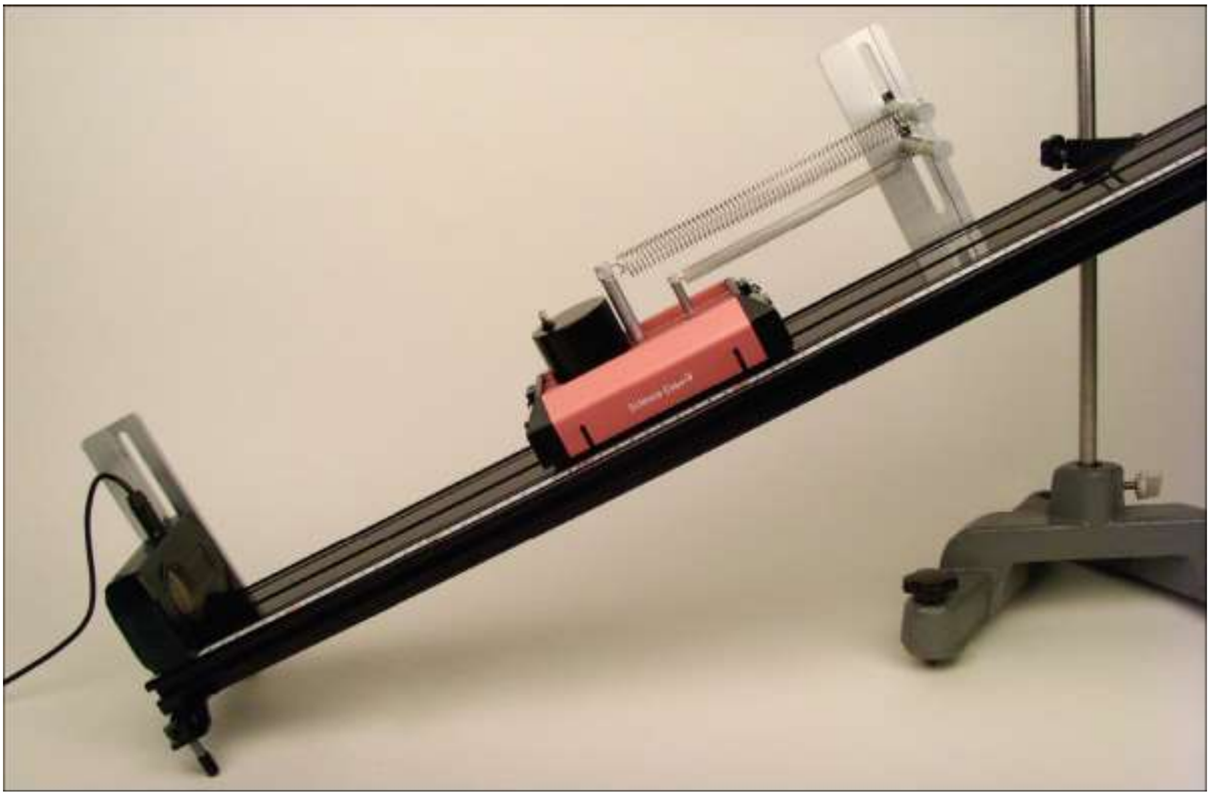
b. As in the explanation of picture 5.5.13, execute the experiment in "Sheet1" of "Oscillation.xls" file.

---

[84] Observe how the results change according to the gradient.

c. Because two springs are connected, input spring K_1's modulus of elasticity to cell E4, and K_2's to cell E5. In cell E6, write the formula "$=E_4*E_5/(E_4+E_5)$"[85].

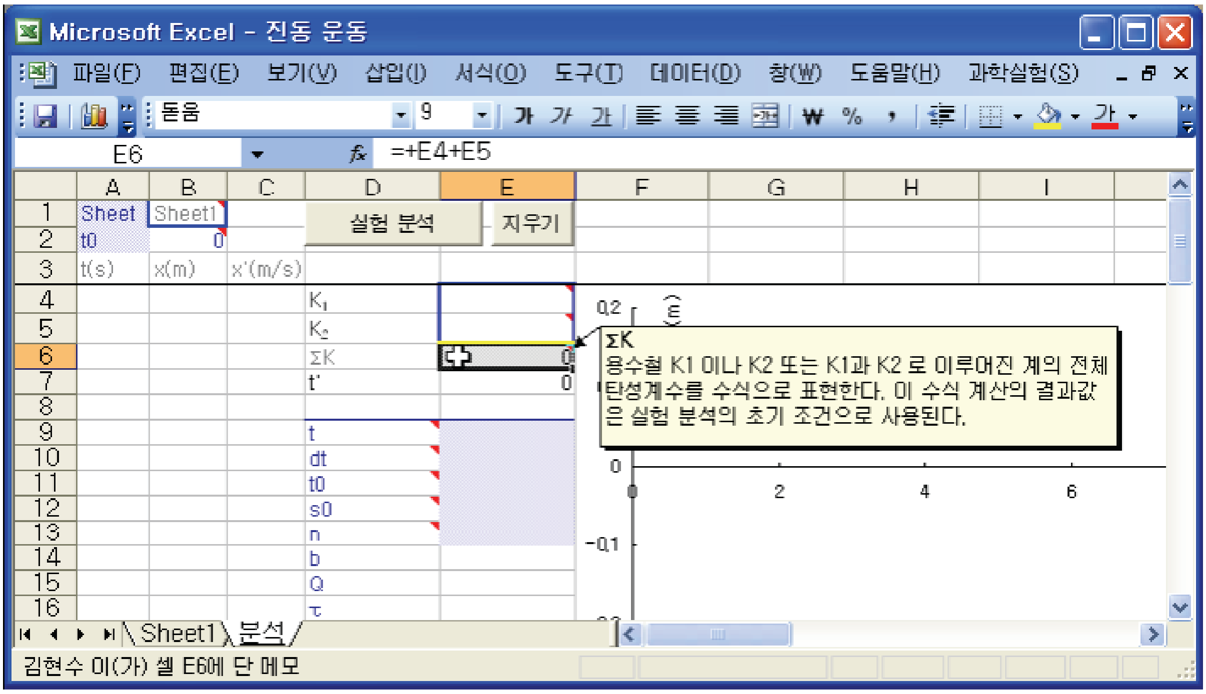
d. Analyze the result by clicking [Experiment Analysis] button in "Analysis" sheet.



Picture 5.5.17 oscillation experiment of a cart that connects two springs in a row[86]

[85] In "Analysis" sheet of "Oscillation.xls" file, cell E6 is set up as formula "$= E_4+ E_5$" to add the value of K_1 and K_2. When the springs are connected in a row, this formula does not fit, so you should modify it. E6 is used the initial value of the system's modulus of elasticity during the analysis process of Excel VBA when clicking [Experiment Analysis].

[86] Change the way of connecting springs variously and execute experiments.



Picture 5.5.18 writing formula of the modulus of elasticity that fits the circumstance of two springs in a row: Input formula “=E_4*E_5/(E_4+ E_5)” to E6.

e. Compare and explain how connecting one spring is different from connecting two springs in a row[87].

[87] When two springs are connected in a row, the total modulus of elasticity can be calculated as $\frac{1}{K}=\frac{1}{K_1}+\frac{1}{K_2}$. Calculate the modulus of elasticity K with the experiment and compare it with the value calculated by the theoretical formula.

.

Experiment Explanation: Damped Oscillation

1. Fill out the table below with the results of experiment analysis.

	Experiment 1	Experiment 2	Experiment 3
b			
Q			
τ			
f_0			
f_1			
β			
$\omega_0^2 - \beta^2$			
m'			
Gradient of slope			
Modulus of elasticity(K)			
Mass of a cart			

Table 5.5.2 results of a cart's oscillation experiment

1. Explain the results in table 5.5.2.

2. How do the results below change according to the gradient of the slope?

   a. Period of oscillation ______________________________

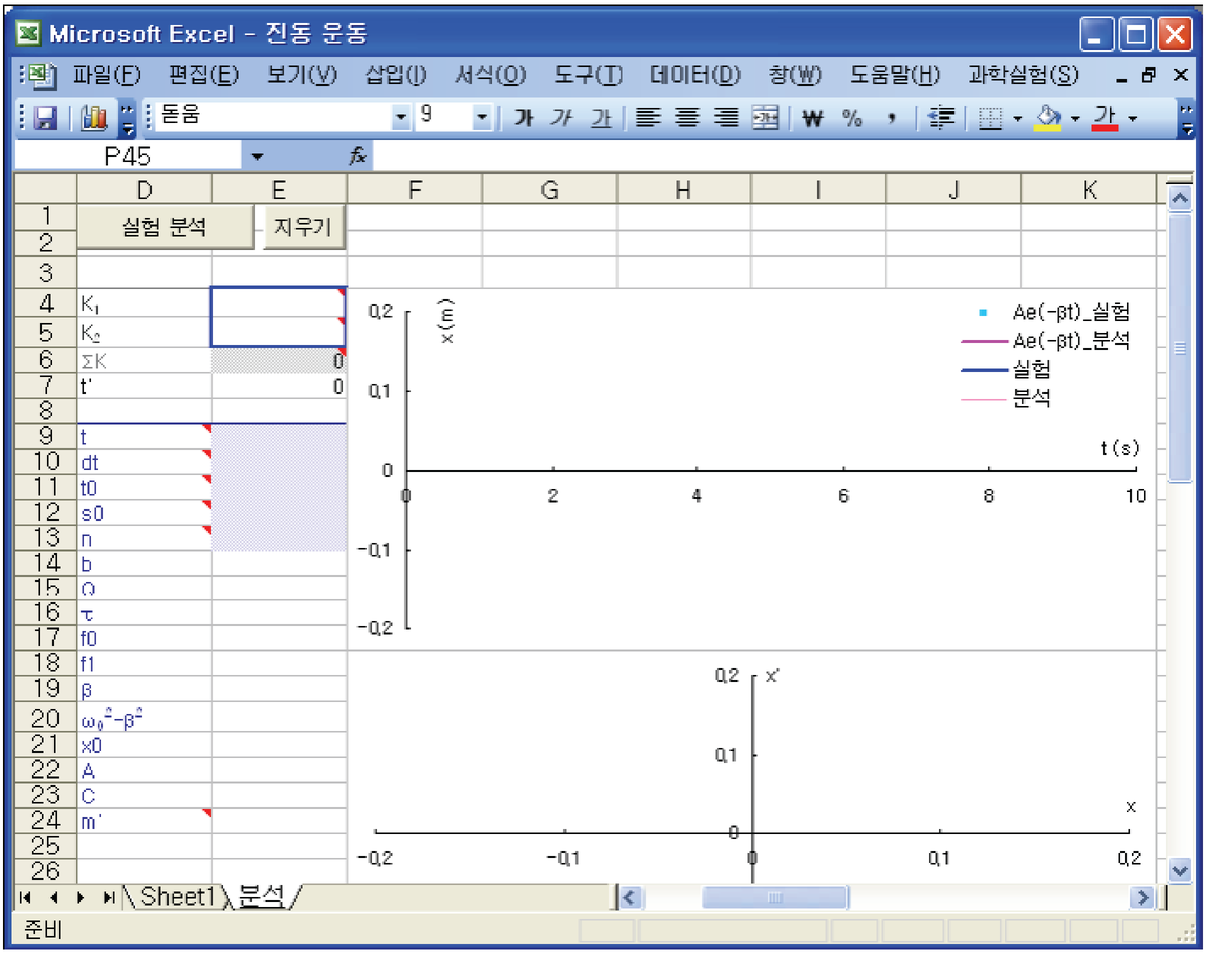
   b. Damping resistance ______________________________

3. How do the results below change according to the mass of the cart?

a. Period of oscillation ______________________________

b. Damping resistance ______________________________

4. Analyze and explain the graph of displacement and time $(x - t)$ and displacement and phase space $(x - \dot{x})$.



Picture 5.5.19 analyzing graphs of displacement, phase space[88]

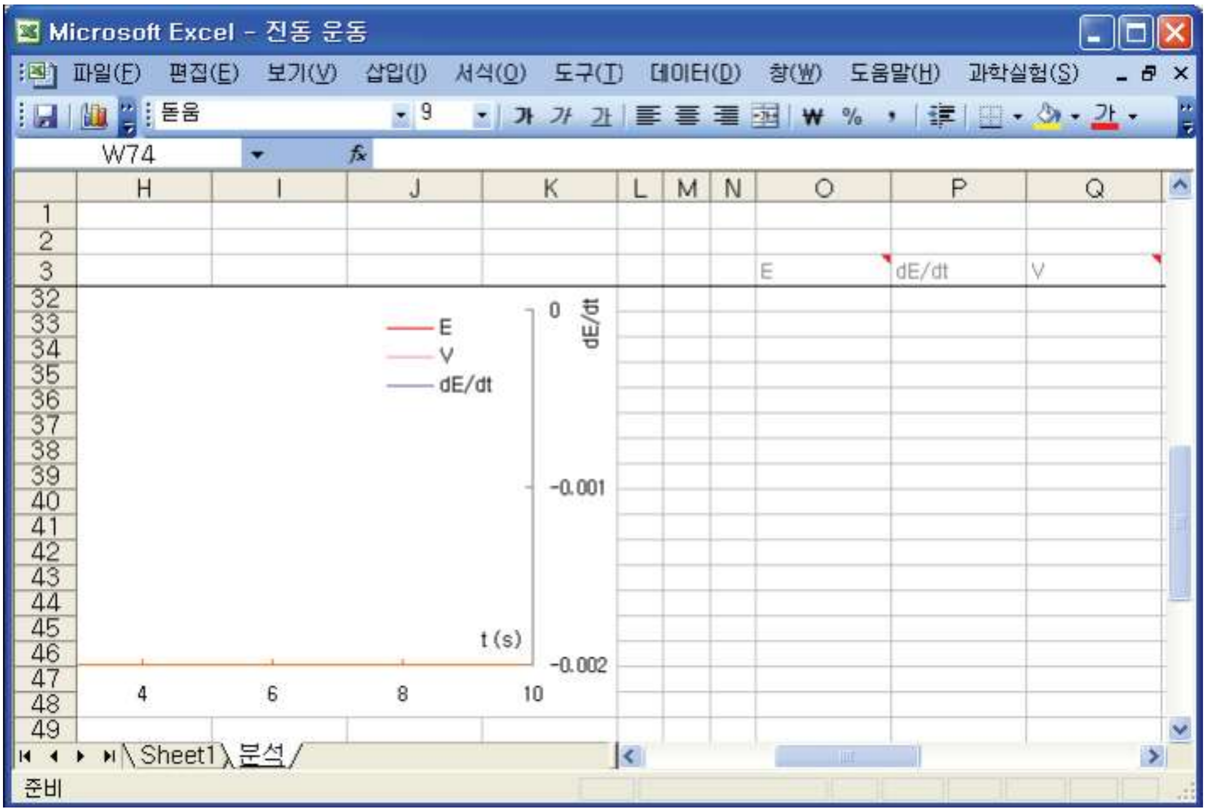
5. How are the things below different according to the mass of the cart and the gradient of the slope?

a. Time constant γ ____________________________

[88] If you click [Experiment Analysis] button in “Analysis” sheet of “Oscillation.xls” file, the analyzed data of time, displacement and velocity will be recorded in column A, B and C. The graph charts of $(x-t)$ and $(x-\dot{x})$, which has been prepared in “Analysis” sheet, will be drawn automatically.

b. Damping constant β ________________________

6. Analyze and explain the graphs of total energy and time (E-t), energy loss ratio and time (dE/dt-t) and the potential energy and time (V-T)[89].



Picture 5.5.20 analyzing E-t graph: If you click [Experiment Analysis] button in "Analysis" sheet, the energy analysis data will be recorded in column O, P and Q.

7. Draw and explain the graphs of velocity and time $(\dot{x} - t)$ and acceleration and time $(\ddot{x} - t)$.

[89] As in "Experiment A", use "Oscillation.xls" file.

Deepened Explanation:

1. As in picture 5.5.17, on a slope, how is the period of oscillation for the two-springs-in-a-row system different from that for one spring system?

2. Explain how the experiment circumstance for two springs connected in series on the horizontal surface, which is as picture 5.5.7, is different from that for two springs connected in a row on the slope, which is as picture 5.5.17.

5.5.4 Experiment Questions

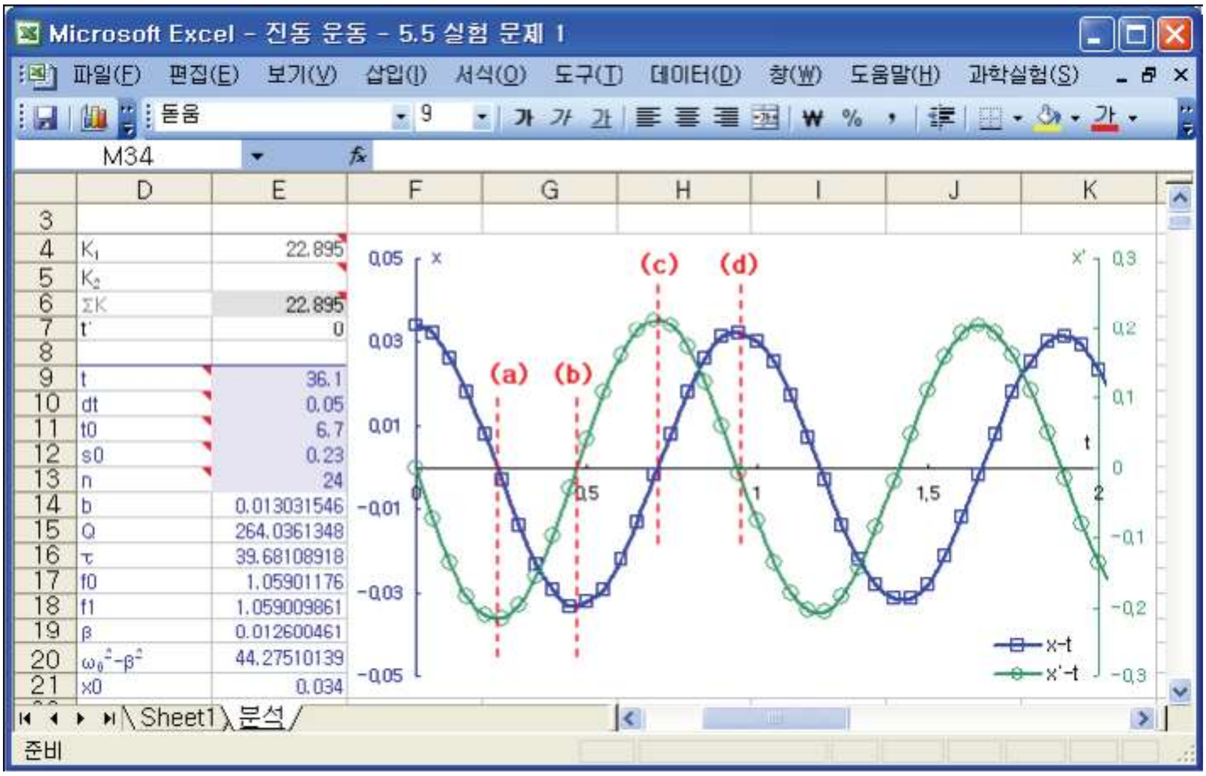
1. Below are the factors concerned with the oscillation of a cart on the horizontal surface as in picture 5.5.1. Based on the experiment results, explain whether each of them is related to the cart's oscillation or not.

a. The cart's mass is related to the frictional force between the cart's wheels and the track's surface and it causes the damping resistance.

b. The cart's period or oscillation is different according to the spring's modulus of elasticity and the cart's mass.

c. The size of the cart's damping resistance affects the period of oscillation and this size can be expressed as damping constant.

2. Picture 5.5.21 is the result and graph of a cart's oscillation on the horizontal surface. Explain the displacement, velocity and acceleration on (a), (b), (c) and (d) points of $x-t$, $\dot{x}-t$ graphs.



Picture 5.5.21 graph of a cart's displacement and velocity: the blue curve is $x-t$ and the green curve is $\dot{x}-t$.

a. When do the displacement, velocity and acceleration become 0 each?

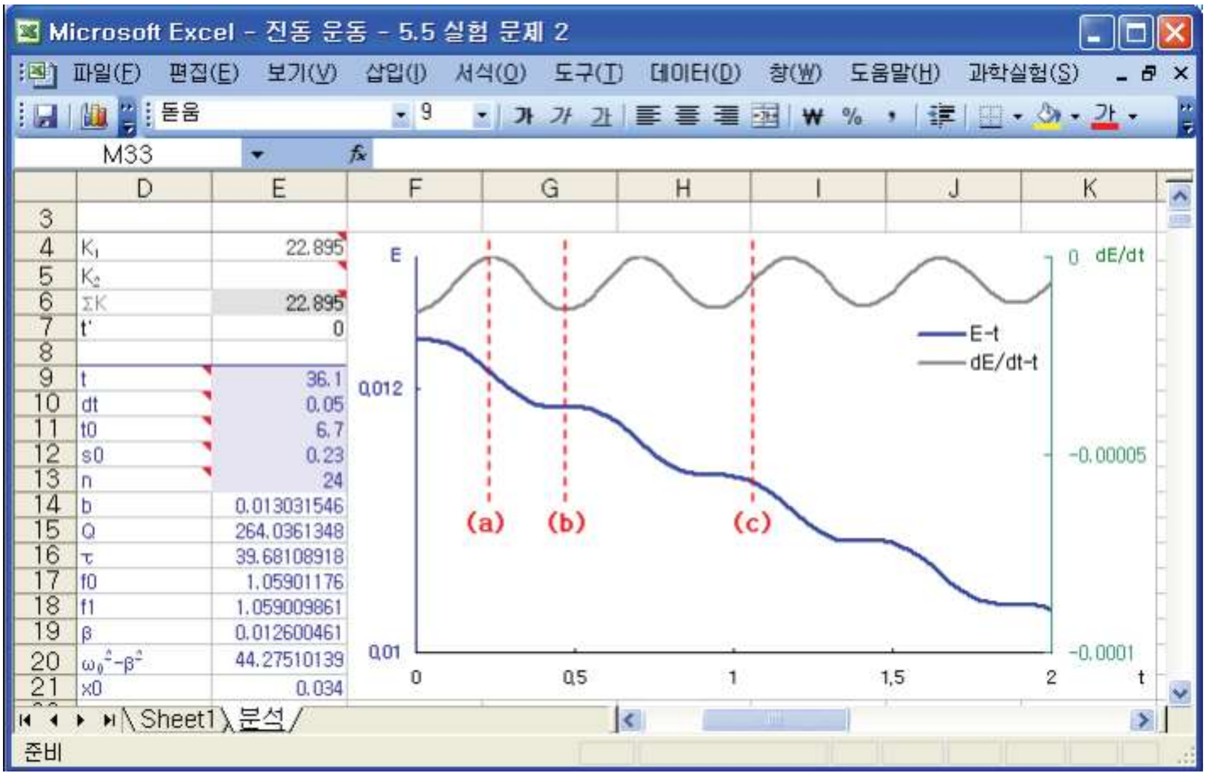
b. In + or −, when is the velocity at the maximum? How about the displacement?

c. In + or −, when is the acceleration[90] at the maximum?

d. From $\dot{x}-t$ graph raw the acceleration and time graph $\ddot{x}-t$. With this graph, explain the force that affects the oscillating cart. Is this force constant? Or is it not?

[90] You can see in the graph that the acceleration is not constant.

3. Picture 5.5.22 is the result and graph of a cart's oscillation on the horizontal surface. In graphs of $E-t, dE/dt-t$, explain the energy and the cart's motion state on (a) and (b) points.



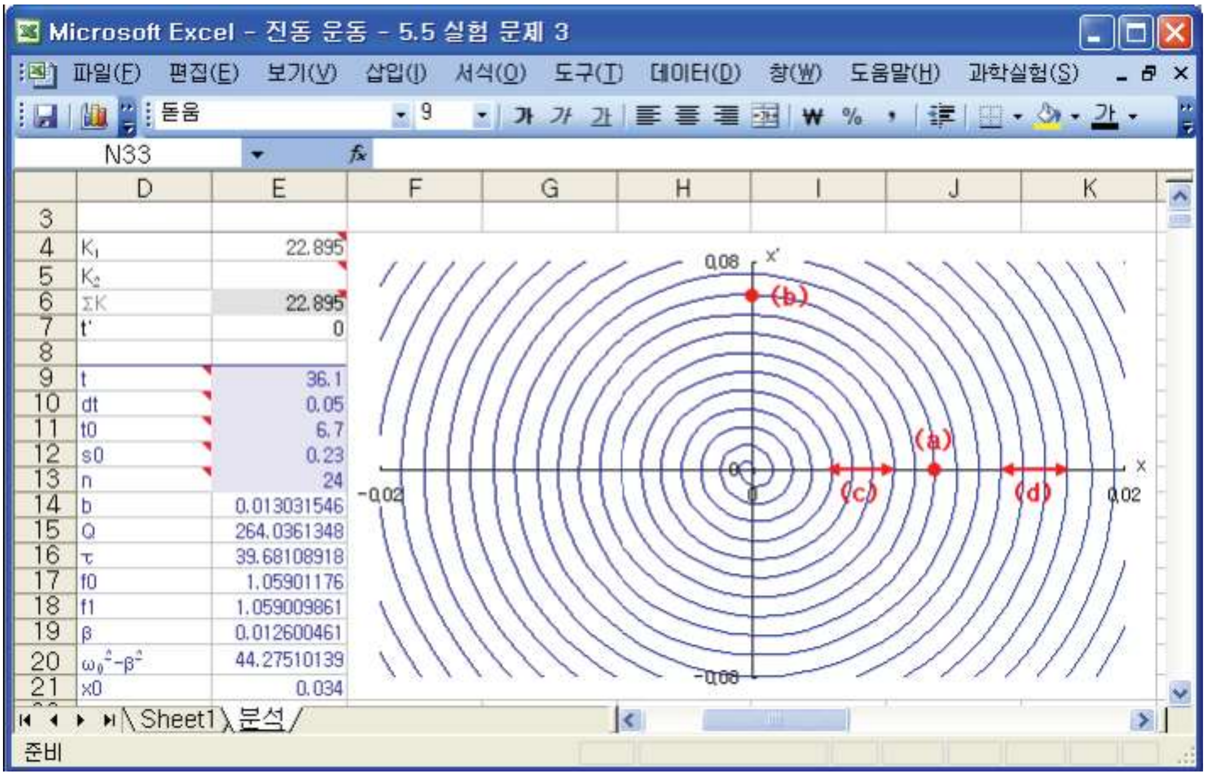
Picture 5.5.22 graph of a cart's energy relationship: the blue curve is $E-t$, and the grey curve is $dE/dt-t$.

a. the energy and motion state on point (a) of $E-t, dE/dt-t$ graph

b. the energy and motion state on point (b) of $E-t, dE/dt-t$ graph

c. the energy and motion state on point (c) of $E-t, dE/dt-t$ graph

4. Picture 5.5.23 is the phase space graph of $x-\dot{x}$, which analyzes the result, displacement and the velocity of a cart's oscillation on the horizontal surface with exponential curve fitting. Explain the things below.



Picture 5.5.23 $x-\dot{x}$ graph of a cart's displacement and velocity[91]

a. the cart's motion state on points (a) and (b) of $x-\dot{x}$ graph

91 The scales of x and $\dot{x}$ are expanded to see parts of the whole motion section.

b. (c) and (d), which represent the displacement's interval Δx, are the same sizes on the graph. What does this fact indicate?

5. Think about how to make the cart's damping resistance bigger on the horizontal surface. What should be done to make critical damping motion? Plan and execute an experiment for this.

6. The motion of a galvanometer's needle[92] is near to the critical damped oscillation. Like this, find out examples that use the damped oscillation around us. Explain the examples below.

a. various shock absorbers concerning to cars

b. motion of sliding or hinged doors that have shock absorbers

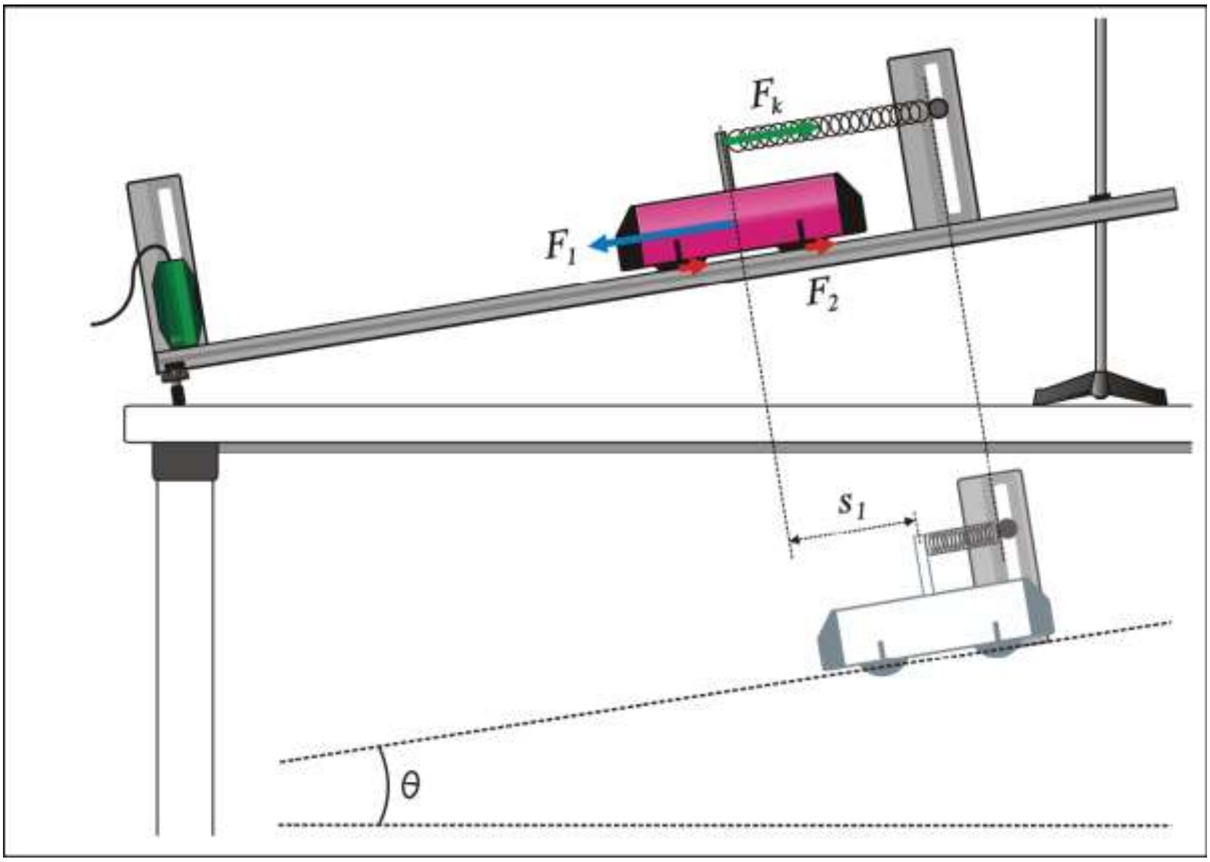
c. uses and principles of various shock absorbers concerning to everyday life

7. With theories and based on the experiment result, explain what factors affect the oscillation period and damping of the cart on the slope.

[92] Galvanometer is the instrument that measures the electric current and it shows the (+) and (-) direction of the electric current so the needle is at the middle when the electric current is 0.

Deepened Questions

1. As in picture 5.5.11[93], set up the equation of motion for the oscillation on the slope. Solve the displacement x and velocity $\dot{x}$ and compare this with the real experiment result.



Picture 5.5.24 oscillation of a cart on the slope: IN picture 5.5.11, the relationships between forces are represented as vector.

Explanation:

When the cart is in equilibrium, concerning the frictional force with the slope, the

[93] The force that affects the cart because of the gravity is different according to the angle θ between the track's slope and the horizontal surface.

force that affects the cart because of the gravity balances with the elasticity.

$$F_g = F_1 - F_2 = (mgsin\theta - \mu mgcos\theta)$$

With this force, when the spring expands as much as the displacement s_1 and it balances with the elasticity, $F_g = -ks_1$.

The relationships for the forces affecting this system are as below.

$$m\ddot{x} + \beta\dot{x} + kx - F_q = 0 \qquad (5.5.1)$$

So the equation of motion can be rewritten as below.

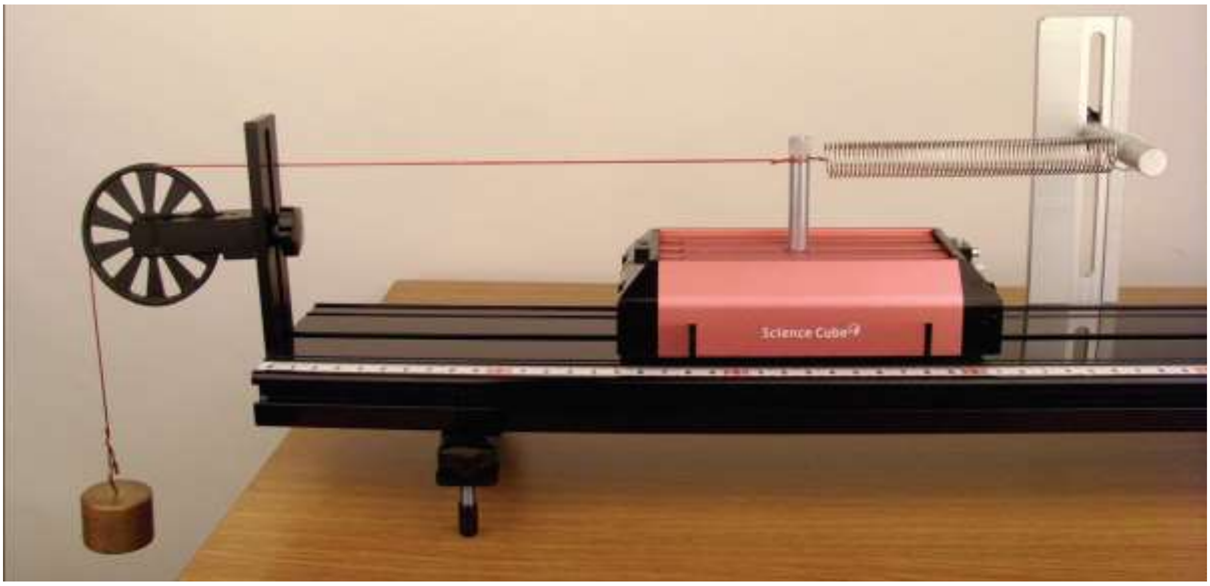
$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x - \omega_0^2 s_1 = 0$$

As in picture 5.5.24, the force affecting the cart is the elasticity and the force concerning to the gravity. So the form of the equation of elasticity can become as formula 5.3.8 and 5.3.9. As in formula 5.3.10 in which the spring's expanded length by the gravity is h_1, in the circumstance of 5.5.21, the expanded length is s_1, so the solution about the displacement x can be solved as follows, which is the same form of formula 5.3.10.

$$x = s_1 + Ae^{-\beta t}\cos\omega t \qquad (5.5.2)$$

In this process, the mass that the spring's motion affects the system has not been applied. However, if the damping constant is calculated with the real experiment, the omitted causes[94] will be expressed.

2. In picture 5.5.25, on the horizontal surface, a pulling spring is hanging at one side of the cart and a pendulum at the other side of the cart using a pulley. Set up the equation of motion for this experiment[95] circumstance and solve the solution. Compare and explain this with other experiments dealt with before.



Picture 5.5.25 oscillation of a cart with a pulling spring on the horizontal surface

Explanation:

[94] The simulations dealt with before are simplified models that omit complex factors concerning to the motion so there are gaps between the real experiment and the theories. That's why the system's mass calculated with the experiment and the cart's mass are different. Theoretically, it's too complicated to consider these small factors and there will be difficulties dealing them with formula.

[95] To measure the cart's motion, measure the cart's location with the motion sensor.

Picture 5.5.26 is for explaining the experiment circumstance of picture 5.5.25. When the cart is in the equilibrium, the force that affects the cart by the gravity balances with the elasticity. When neglecting other forces' influences on the system, the force that affects the cart with the pendulum is the total weight of the pendulum mg.

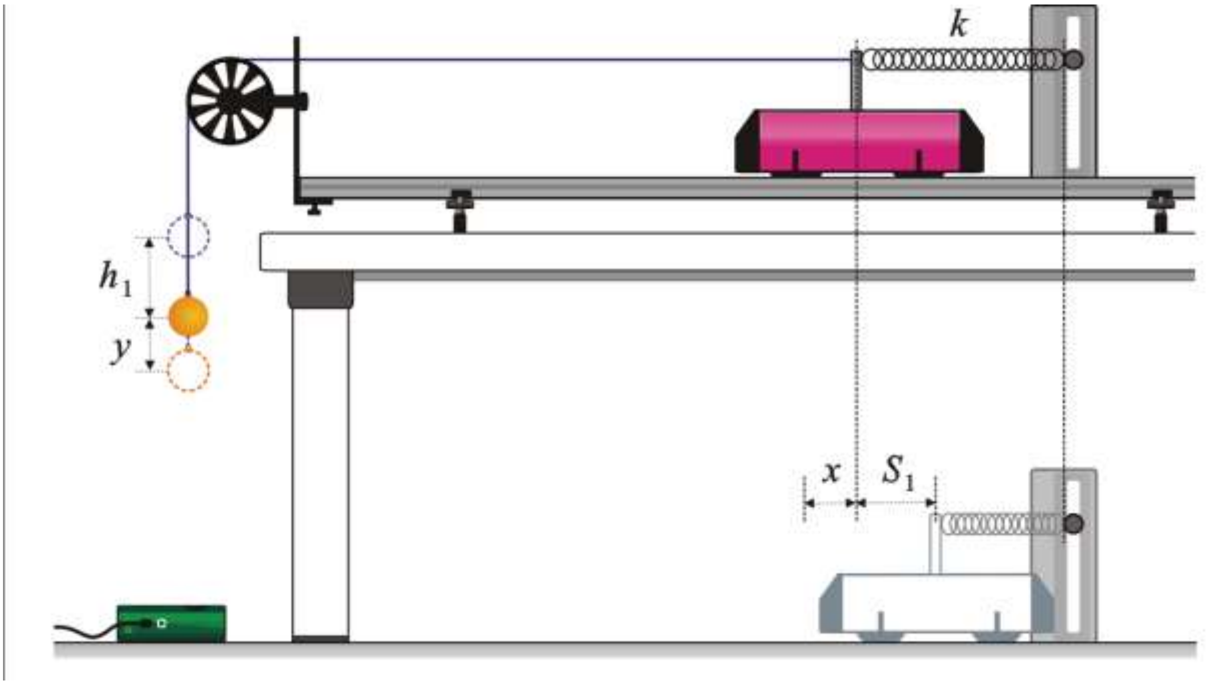
With this force, when the spring is expanded as much as the displacement s_1 and it balances with the elasticity $-ks_1$, it can be as below.

$$mg = |-ks_1| = |-kh_1|$$

So the total force that affects this system is like below.

$$m\ddot{x} + \beta\dot{x} + kx - mg = 0$$

This is the same form as formula 5.5.1, which was solved above.



Picture 5.5.26 oscillation of a cart on the horizontal surface: the experiment circumstance of picture 5.5.25

When you see picture 5.5.26, when the cart is expended as s_1, the pendulum is lowered as h_1 and they balance with each other, so when the cart's displacement changes as much as x, the pendulum's height will change as y. The location and velocity can be solved with the processes dealt with above so you can try solving them, too.

5.6.

Experiment: Forced Vibration

5.6.1. Experiment Outline

Calculate frequency and amplitude near the resonance in the forced vibration experiment of a mass-spring system consisted of a cart and a spring. Change the mass and the damping of the system to understand the oscillation's characteristics according to the physical circumstances of the forced vibration.

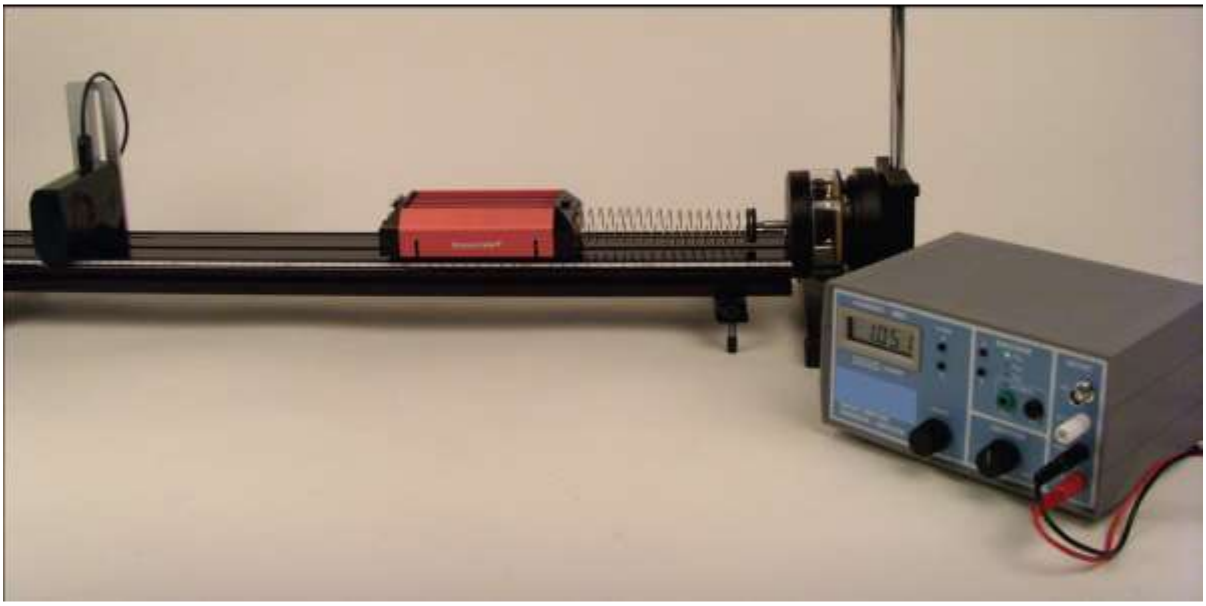
Goal

Understanding the characteristics of a mass-spring system's forced oscillation

Required Equipments

Motion sensor	1	Banana plug	2
Spring (pushing 1, pulling 2)		Pendulum (500g)	1
Mechanical waver driver	1	Magnet (neodymium)	6
Sine function generator	1	Double stick tape	1
Cart	1	Holding tape	1
Track	1		
Pendulums (50g)	1		
Electronic scale (500g)	1		

5.6.2 Experiment: Forced Vibration of a Cart That Gets Damping Resistance



Picture 5.6.1 forced vibration experiment of a system in a damped oscillation: drive the mechanical waver driver with the function generator and make the mass-spring system in a forced vibration.

Experiment Prediction: Amplitude according to the Damping Resistance of a Cart

1. As in picture 5.6.1, consider the case when a cart and pulling-pushing spring system vibrates forcedly by getting a consistent force of a sine function form with the mechanical waver driver.

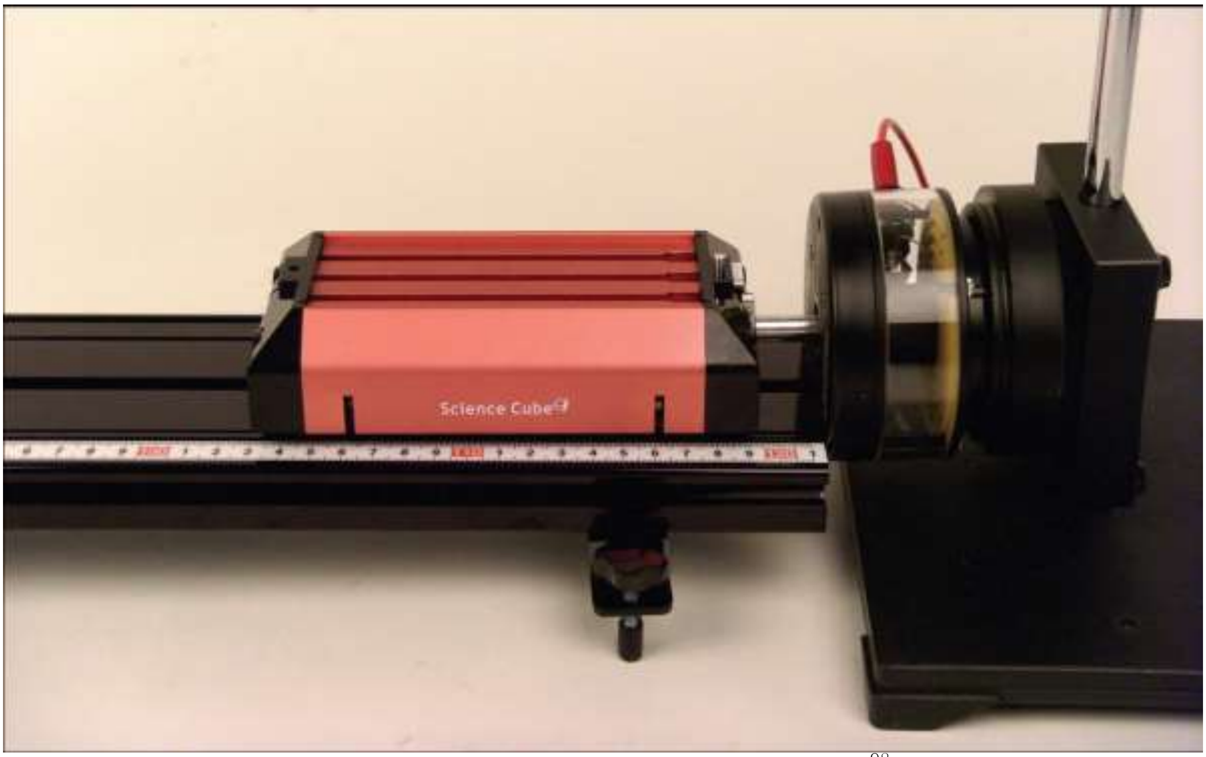
   a. How does the resonance frequency change as the cart's mass gets smaller?

   b. How does the amplitude change as the damping resistance gets bigger?

   c. Draw the angular frequency graph ω/ω_0 and the amplitude graph A/A_0 when the cart's damping resistance is 0.1, 0.5 and 0.7. How does the graph change according to the damping resistance?

   d. Guess how to find the resonance frequency when you do not know the cart's mass and the spring's modulus of elasticity.

Experiment Process A: Measuring the Amplitude of a Mechanical Waver Driver



Picture 5.6.2 measuring the amplitude of a mechanical waver driver[98]

1. Prepare the experiment as follows.

   a. Measure the cart's mass with the electronic scale. Measure the spring's modulus of elasticity in advance.

   b. As in picture 5.6.2, put the cart on the track and set up the motion sensor on the track so that it can measure the cart's location.

   c. Turn on the function generator and set up the amplitude button in the middle. Fix this button with the holding tape.

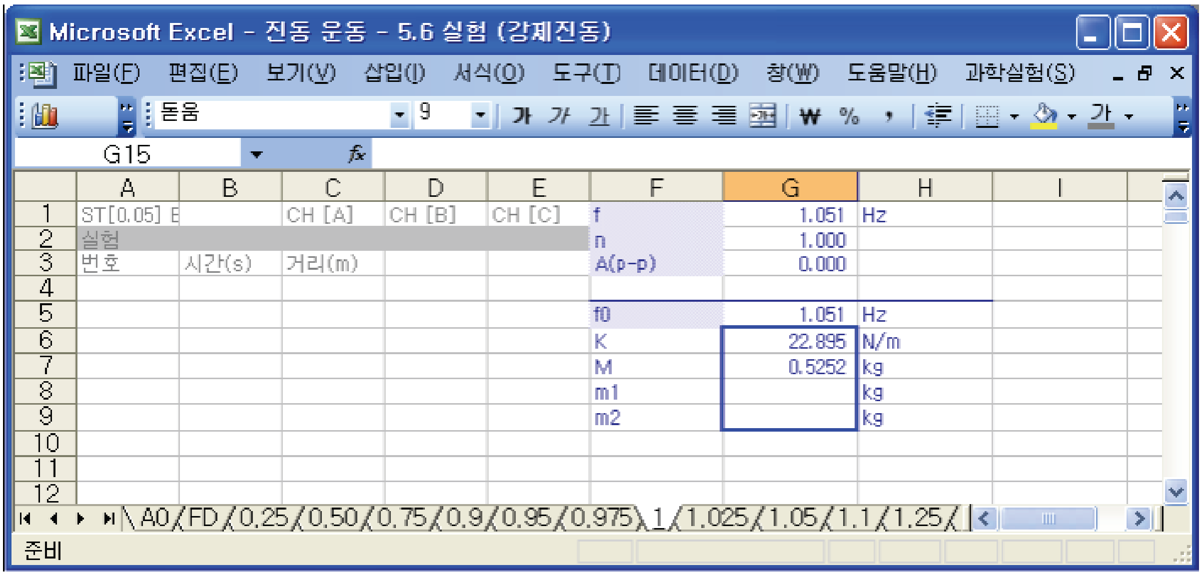
   d. As in picture 5.6.2, connect the cart's right end with the mechanical waver driver using double stick tape.

[98] With the double stick tape, attach the horizontal oscillation pole of the mechanical waver driver to the cart.

2. Connect the sensor and the computer and execute the experiment with “Oscillation (Forced Vibration).xls” file.

   a. Open [Science Cube]-[Experiment Setting] window in the worksheet and set up the measuring interval as 0.05 second, and the experiment time as 20 seconds.

   b. As in picture 5.6.3, in sheet “1” of the workbook, input the modulus of elasticity K, the cart’s mass M and the additional mass m1 or m2 to cell G6, G7, G8 and G9.



Picture 5.6.3 experiment in “Oscillation (Forced Vibration).xls” of Excel: input the initial conditions to cell G6, G7, G8 and G9 of sheet “1”

   c. Open sheet “A0”. Read the frequency value automatically recorded[99] in cell G1 and turn the frequency button of function generator and adjust it to this value.

   d. Connect (+) and (−) terminals of the function generator to the (+) and (−) terminals of the mechanical waver driver using banana plug[100]. After this, the cart will start oscillating.

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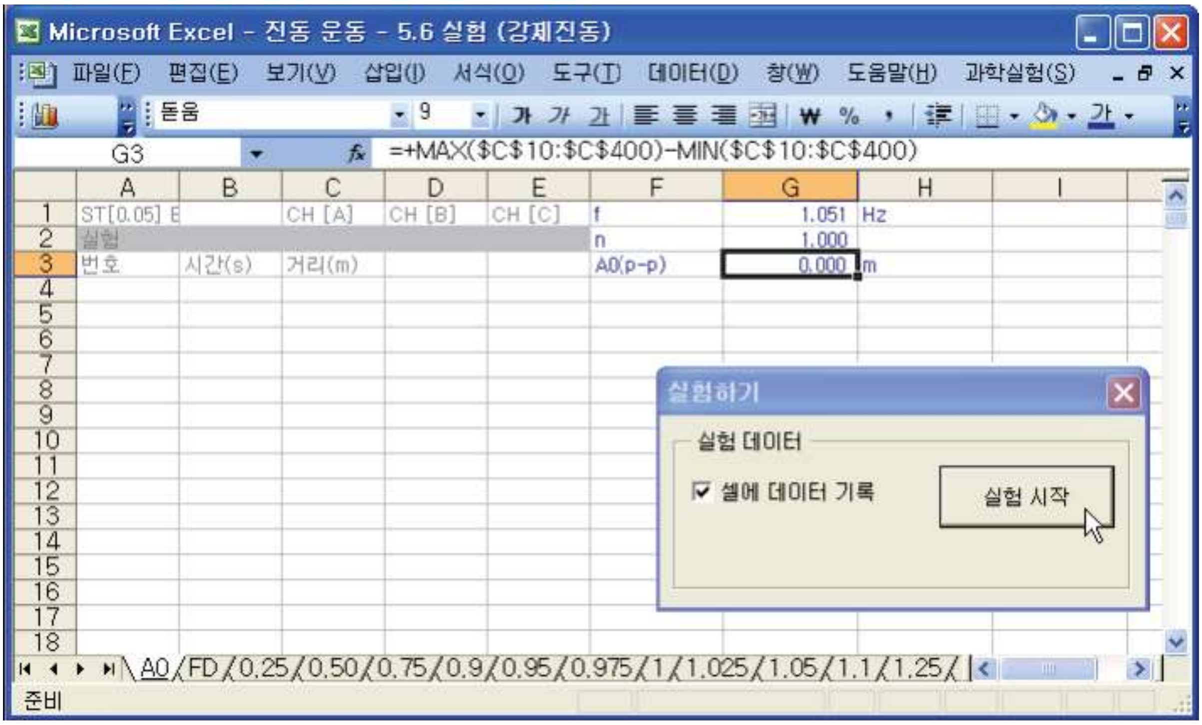
[99] This value is the natural frequency calculated by inputting the modulus of elastic K and mass m, m1 and m2 in sheet “1” and is recorded in cell G1 of sheet “1”, “A0” and “FD”.

[100] The banana plug is a cable that is used with the function generator.

e. While the cart is oscillating, open [Science Cube]-[Experiment] window in the worksheet and click the [Start Experiment] button. If you click it, the experimental data will be collected within the sheet of workbook.



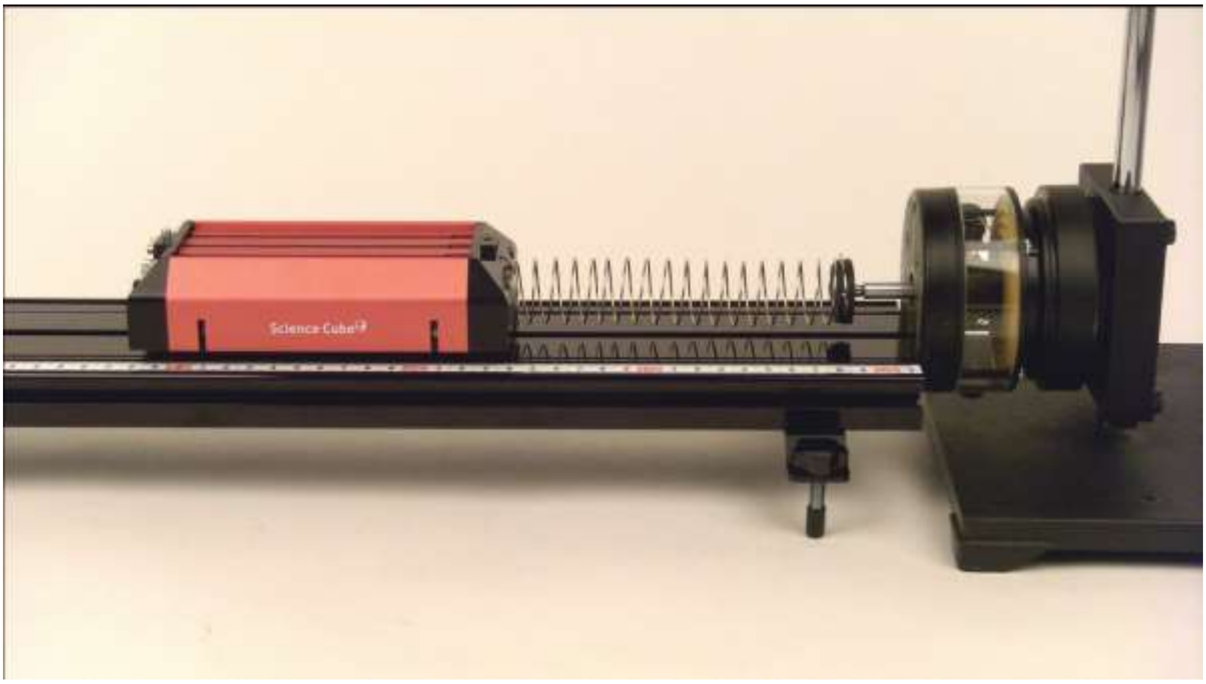
Picture 5.6.4 experiment in "Oscillation (Forced Vibration).xls" of Excel: calculating the P-P amplitude A_0 of the mechanical waver driver in sheet "A0"[101]

f. As in picture 5.6.4, 20 seconds after the experiment's beginning, stop the experiment and read the calculated value in cell 3 of sheet "A0" as the P-P amplitude of the mechanical waver driver A_0.

g. This experiment process A is connected to the next experiment process B.

[101] Calculate the P-P amplitude with the formula "=MAX($C$10:$C$400)-MIN(c10:$C$400)" which is recorded in cell G3.

Experiment Process B: Measuring the Amplitude in the Frequency Area near the Resonance



Picture 5.6.5 making forced vibration of a cart with a mechanical waver driver

1. Experiment process B follows experiment process A. After the setup is done as a, b, and c of experiment process 1, as in picture 5.6.5, attach a spring to the right end of the cart and connect the spring's end to the mechanical waver driver.

2. Execute the experiment with "Oscillation (Forced Vibration).xls" file opened[102].

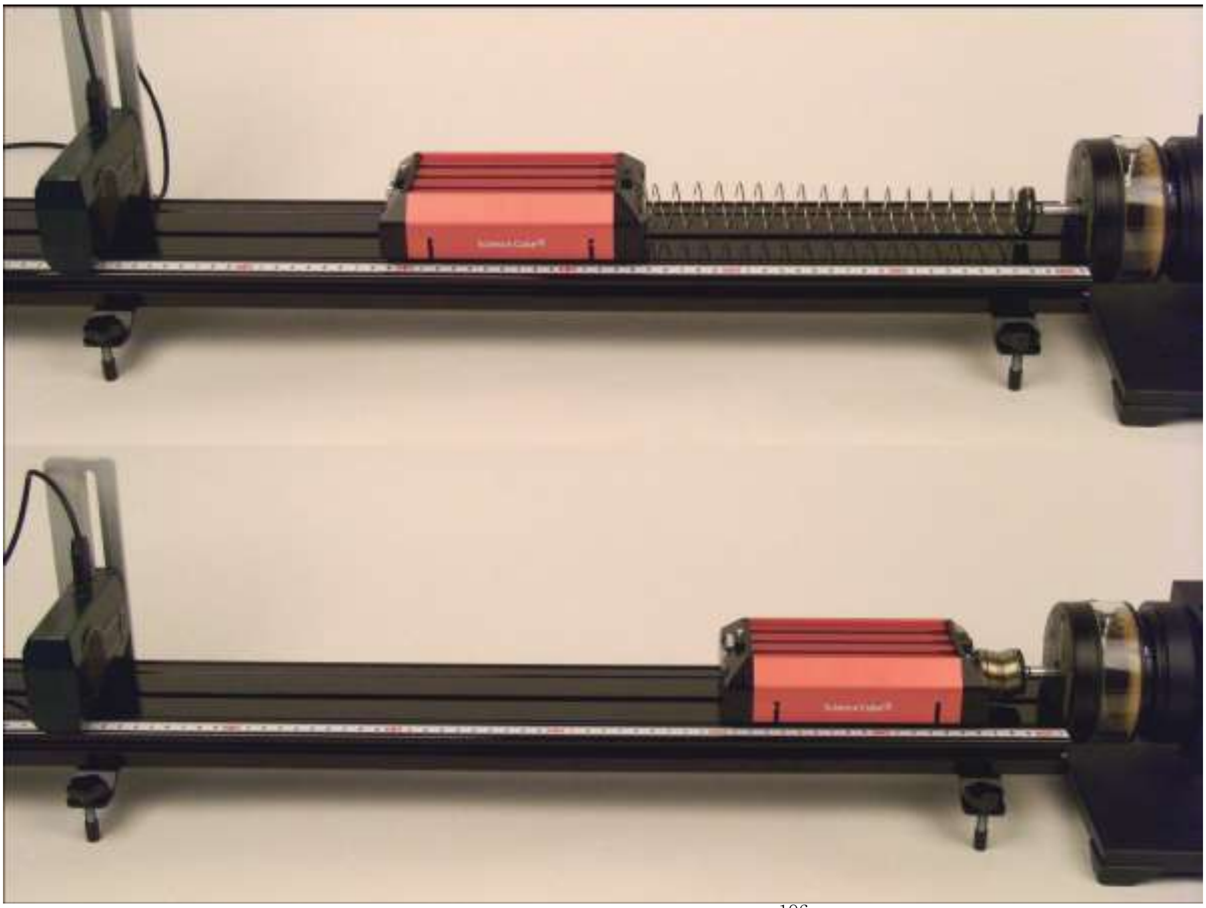
   a. Open [Science Cube]-[Experiment Setting] window in the worksheet and set up the measuring interval as 0.05 second, and the experiment time as 20 seconds.

   b. Open sheet "0.25". Check out whether the value of n is recorded in cell G2 as 0.25[103].

[102] Continue the experiment after experiment process A.
[103] The value of n is recorded in advance in "Oscillation(Forced Vibration).xls" file.

Read the value of frequency recorded automatically in cell G1 and turn the frequency setting button of the function generator to adjust it to this value.

c. Connect (+) and (-) terminals of the function generator to the (+) and (-) terminals of the mechanical waver driver using banana plug[104]. After this, the cart will start oscillating. After it starts oscillating, when it passes the transient state[105] and reaches the normal state, continue the next step d.



Picture 5.6.6 a cart forcedly vibrating near the resonance[106]

[104] The banana plug is a cable that is used with the function generator.

[105] As in case of 0.975 and 1.025, when it is near the resonance, the P-P amplitude of the cart becomes the maximum and the spring is no longer stretched or compressed. When n is near 1 as $n<<1$ or $n>>1$, the unstable transient state might last for a long time from 10 to 30 seconds.

[106] The picture shows that the cart is on the P-P amplitude location.

d. While the cart is oscillating, open [Science Cube]-[Experiment] window in the worksheet and click the [Start Experiment] button. If you click it, the experimental data will be collected within the sheet of workbook.

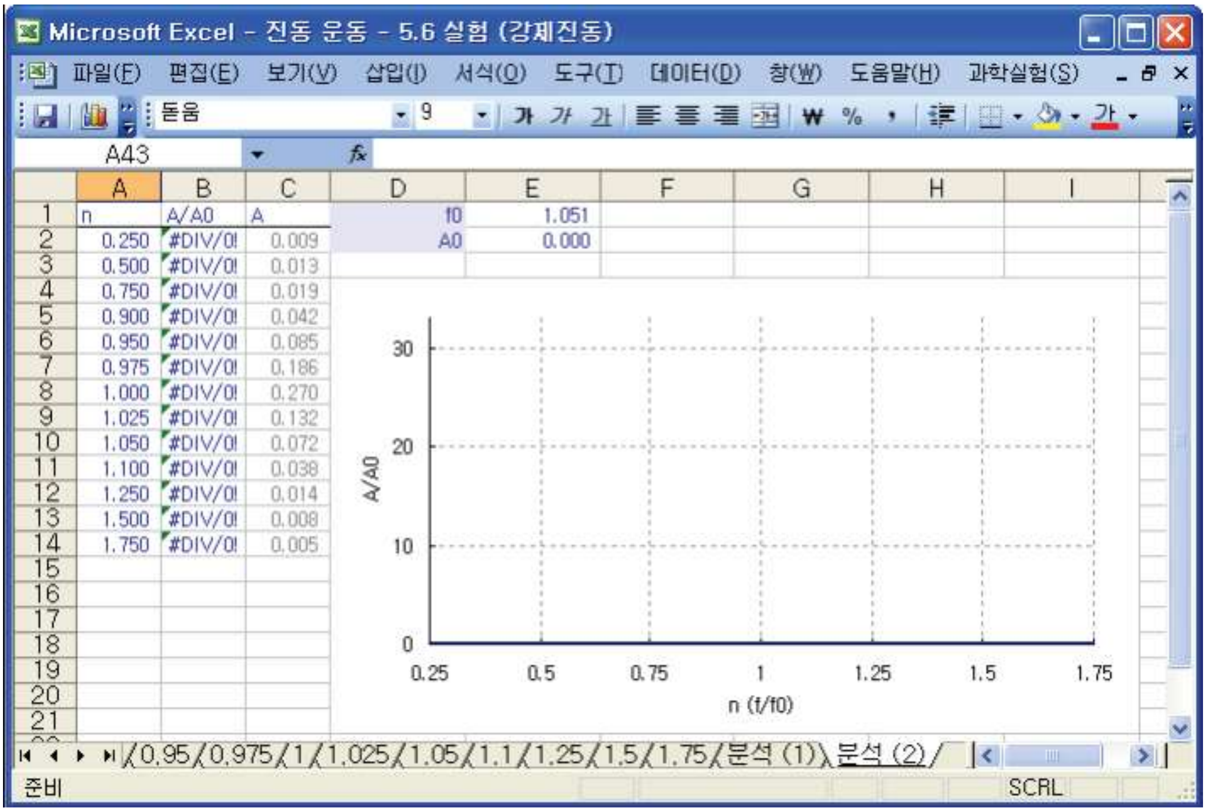
Picture 5.6.7 experiment in “Oscillation (Forced Vibration).xls” of Excel: calculating P-P amplitude in sheet “0.25”

f. After 20 seconds, stop experiment and check out P-P amplitude A(p-p) calculated by formula and recorded in cell G3 of sheet “0.25”.

g. In case of n=0.5, 0.75, 0.9, 0.98, 1, 1.05, 1.1, 1.25, 1.5 and 1.75, move within the prepared sheets “0.5”, “0.75”, “0.9”, “0.95”. “1”, “1.05”, “1.1”, “1.25”, “1.5” and “1.75”, and repeat the processes from b to f.

h. After finishing the experiment process g, open sheet “Analysis (2)” and check out the graph of frequency f/f_0 and of amplitude A/A_0 [107].

[107] The graph has been drawn already in sheet “Analysis (2)” of “Oscillation (Forced Vibration).xls” file. When you finish the experiment as in picture 5.6.7, the graph is drawn according to the values calculated by the formula in column A, B, and C.



Picture 5.6.8 experiment in “Oscillation (Forced Vibration).xls” of Excel: check out the graph of frequency f/f_0 and of amplitude A/A_0 in sheet “Analysis (2)”[108]

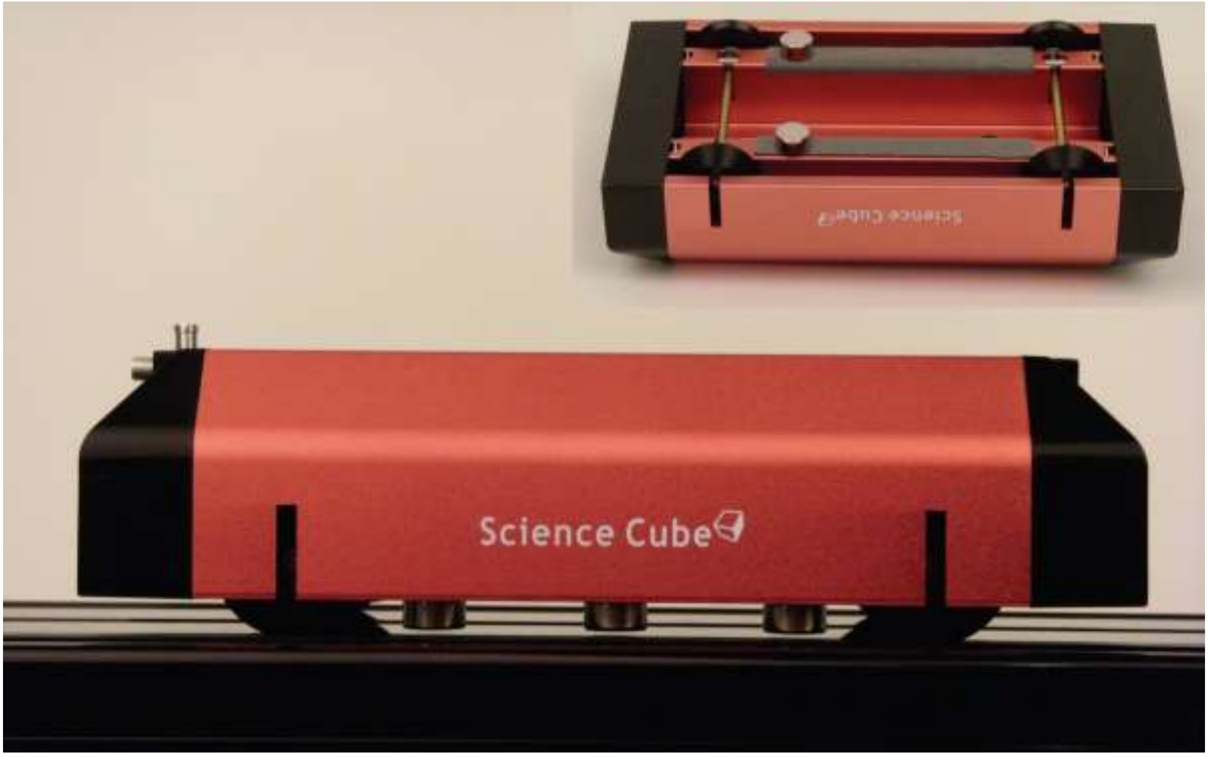
Experiment Process C: Experiment with Different Damping Resistances

1. As in picture 5.6.9, attach a magnet to the bottom of a cart and make the damping resistance bigger.

2. Execute the experiment process A and B by changing the number of the magnet as 1, 2, and 4, and calculate the graph of frequency f/f_0 and of amplitude A/A_0.

3. Repeat the process 1 and 2 above by changing the cart’s mass. Process 1 and 2 include the process of continuing the process A and B.

[108] Before the experiment, the values of column B and cell E1 of sheet “Analysis (2)” are all “#DIV/0!”. After finishing the experiment from sheet 0.25 to 1.75, these values are calculated as the results of the experiment and recorded in the corresponding cells. The graph is also drawn by the values of column A and B.



Picture 5.6.9 experiment with different damping resistances: make the damping resistance bigger by attaching magnets[109] to the cart

Deepened Experiment: The Reaction of a Cart in Transient State

1. Within the range of 1<n<3, Change the frequency of the cart and execute the experiment.

   a. Collect data for 120 seconds[110] when the cart is not moving.

   b. Draw $x-t$ graph and explain it.

[109] When attaching magnets, use the double stick tape and make sure the magnets don't touch the track. According to the number of the magnet the damping resistance changes, so change the number of magnet as 1, 2, 4 or 6 and execute the experiment.

[110] This is for the transient state, so you can change the measuring time longer or shorter according to the circumstances.

Experiment Explanation: Forced Vibration

1. Write the experiment analysis results in the table.

f_0	
A_0(p−p)	
K	
M	
m_1, m_2	

Table 5.6.1 result of a cart's forced vibration experiment (1) : initial conditions of experiment[111]

	Damping Magnet	n(β)=1	2	4
	0.25			
	0.5			
	0.75			
	0.9			
	0.95			
	0.975			
n=	1.0			
	1.025			
	1.05			
	1.1			
	1.25			
	1.5			
	1.75			

Table 5.6.2 result of a cart's forced vibration experiment (2): result of measuring P-P amplitude A(p-p) per $n = f/f_0$

111 m_1 and m_2 are the additional masses attached to the cart such as the double stick tape, magnets and so on.

2. Explain the result in table 5.6.2. When the damping resistance gets bigger, how does the maximum amplitude of a cart change near the resonance (n=1)?

3. Express graph of $n = f/f_0$ and P-P amplitude A(p-p) synthetically[112] when the damping magnet is 1, 2 and 4, and explain it.

4. Are the cart's natural frequency (f_0) and resonant frequency (f_n) hugely different? Or are they not? What makes the cart's natural frequency change? Explain this with the experiment results.

5. Does the cart's phase A get near to F/K within the range of n<<1? Explain this.

6. Express graph of $n = f/f_0$ and P-P amplitude A(p-p) synthetically when the cart's mass is different and explain it.

Deepened Explanation:

1. How is the cart's motion in the transient state? How does the $x-t$ graph's shape change according to the value of $f - f_0$[113]?

[112] When you finish the experiment in "Oscillation(Forced Vibration).xls" file, the synthetic graph of $n = f/f_0$ and P-P amplitude A(p-p), which has been made already in sheet "Analysis (2)", will be drawn based upon the experiment results.

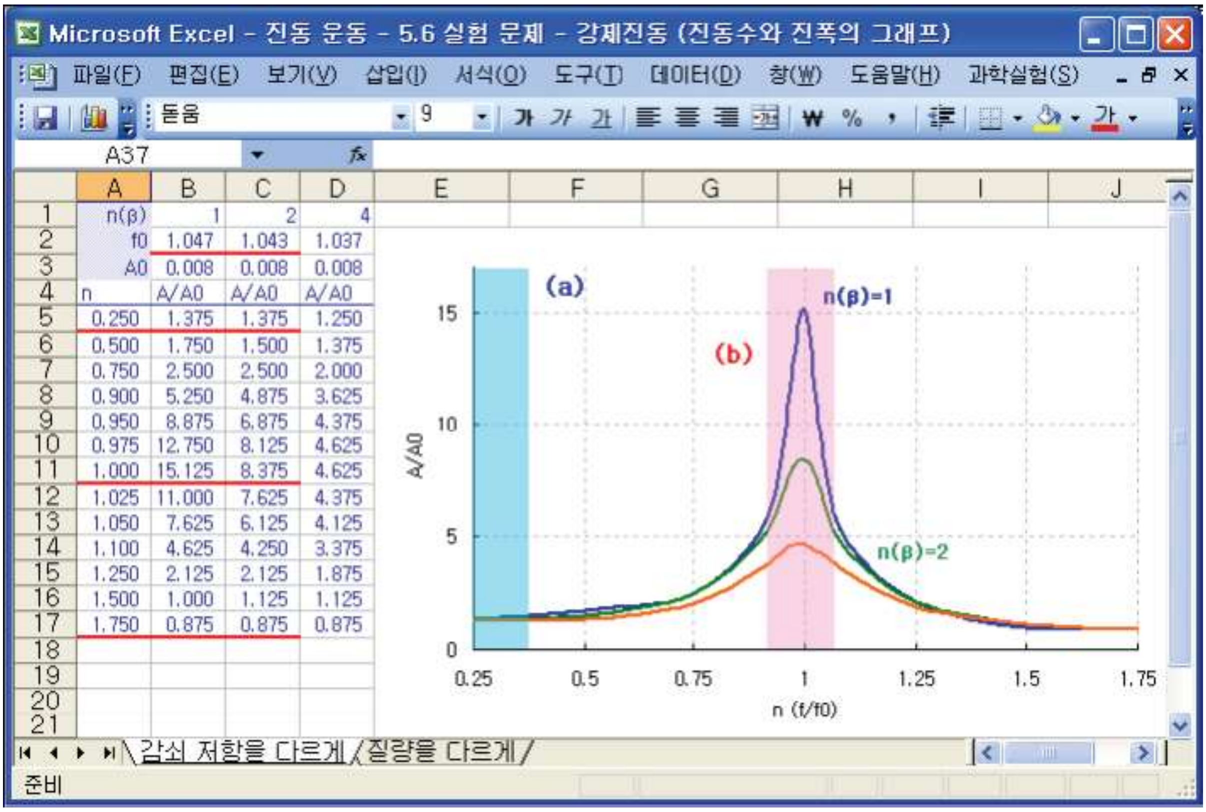
[113] $f - f_0 = (n-1) f_0$

5.6.3 Experiment Question

1. Picture 5.6.10 is the results of forced vibration by changing the damping resistance of carts that have masses of 0.529kg and 0.532kg using a spring that has the modulus of elasticity 22.895N/m. Based on the table 5.6.3, Explain the questions below about the cart's motion.

		A/A_0 of f_0=1.047	A/A_0 of f_0=1.043
	0.25	1.375	1.375
n=	1.0	15.125	8.375
	1.75	0.875	0.875

Table 5.6.3 result of a cart's forced vibration experiment



Picture 5.6.10 graph of n and A(p-p) according to the result of a cart's forced vibration experiment[114]

[114] The Excel workbook file of picture 5.6.10 can be downloaded at www.sciencecube.com.

a. According to the experiment result of table 5.6.3, the 0.003kg difference between two cart's masses causes the natural frequency's 0.004Hz difference. Then does the difference of mass cause the difference of the peaks' heights in graphs of picture 5.6.10? Or does it not?

b. In graph of picture 5.6.10, explain Q-constant and damping resistance. In (b) of the graph, does the damping resistance decide the peak's height?

c. Two carts' A(p-p) is 1.375 when n=0.25. As n gets smaller, to which value does this value get nearer? How big is the value?

d. Two carts' A(p-p) is 0.875 when n=1.75. As n gets bigger, to which value does this value get nearer? How big is the value?

2. To make the 0.525 ± 0.0001kg cart's damping resistance changed, a magnet of 0.0029 ± 0.00001kg is used. When the number of magnet is changed, the cart's mass and the value of natural frequency are changed. Then, how will $n = f / f_0$ and P-P amplitude A(p-p) change?

a. Does the effect of magnet's damping resistance influence greatly to determine the graph's peak shape?

b. Does it influence greatly to determine the graph's peak shape that the cart's mass changes little because of the magnet's mass? Then, which part is it in the picture 5.6.10 above?